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# QUASILINEAR ELLIPTIC EQUATIONS WITH MORREY DATA

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**Abstract**

We obtain global essential boundedness and Hölder continuity of the weak solutions to quasilinear elliptic equations in divergence form with data lying in Morrey spaces.

**Key words:** quasilinear elliptic operator, discontinuous coefficients, Morrey space, essential boundedness, Hölder continuity

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**1. Introduction and hypotheses.** The general goal of the present paper is to announce our recent results regarding global boundedness and Hölder continuity of the weak solutions to quasilinear divergence form elliptic equations with data lying in Morrey spaces. Precisely, we consider the Dirichlet problem

$$\begin{cases} u \in W_0^{1,m}(\Omega), \quad m > 1 \\ \operatorname{div}\big(\boldsymbol{a}(x,u,Du)\big) = b(x,u,Du) \quad \text{weakly in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq m$, is a bounded domain and the nonlinearities are given by the Carathéodory maps $\boldsymbol{a}(x,z,\xi)\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $b(x,z,\xi)\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, where $\boldsymbol{a}(x,z,\xi) = \big(a^1(x,z,\xi),\dots,a^n(x,z,\xi)\big)$.

In order to list the assumptions on the data of (1) we need to recall the definition of the Morrey spaces. Set $B_\rho(y)$ for the $n$-dimensional ball centred at a point $y$ and of radius $\rho$. Given $p \in [1, \infty)$ and $\lambda \in [0, n]$, a function $u \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|u\|_{L^{p,\lambda}(\Omega)} := \sup_{y\in\Omega,\, \rho\in(0,\operatorname{diam}\Omega)} \left( \rho^{-\lambda} \int_{B_\rho(y)\cap\Omega} |u(x)|^p \, dx \right)^{1/p} < \infty.$$

This quantity defines a norm and $L^{p,\lambda}(\Omega)$ is a Banach space. In the borderline cases $\lambda = 0$ and $\lambda = n$ one has $L^{p,0}(\Omega) \equiv L^p(\Omega)$ and $L^{p,n}(\Omega) \equiv L^\infty(\Omega)$, respectively.

Throughout the note we suppose that the data of (1) satisfy:

• *Controlled growth conditions*: There exist a constant $\Lambda > 0$ and non-negative functions $\varphi$ and $\psi$ such that

$$(2) \qquad \begin{cases} |\boldsymbol{a}(x,z,\xi)| \le \Lambda \left( \varphi(x) + |z|^{\frac{n(m-1)}{n-m}} + |\xi|^{m-1} \right), \\[2ex] |b(x,z,\xi)| \le \Lambda \left( \psi(x) + |z|^{\frac{n(m-1)+m}{n-m}} + |\xi|^{m-1+\frac{m}{n}} \right) \end{cases}$$

for a.a. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, where

$$(3) \qquad \begin{cases} \varphi \in L^{p,\lambda}(\Omega) \quad \text{with } p > \dfrac{m}{m-1}, & \lambda \in (0,n),\ (m-1)p + \lambda > n, \\[2ex] \psi \in L^{q,\mu}(\Omega) \quad \text{with } q > \dfrac{nm}{n(m-1)+m}, & \mu \in (0,n),\ mq + \mu > n. \end{cases}$$

In the particular case $n = m$, $\boldsymbol{a}(x,z,\xi)$ and $b(x,z,\xi)$ may grow with arbitrary positive power of $|z|$, while $b(x,z,\xi)$ has $m$-th gradient growth.

• *Coercivity of the differential operator*: There exists a constant $\gamma > 0$ such that

$$(4) \qquad \boldsymbol{a}(x,z,\xi) \cdot \xi \ge \gamma |\xi|^m - \Lambda |z|^{\frac{nm}{n-m}} - \Lambda \big(\varphi(x)\big)^{\frac{m}{m-1}}$$

for a.a. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

• *The boundary of $\Omega$ satisfies the $(A)$-property* (see [1]) which requires that for each $x \in \overline{\Omega}$ the Lebesgue measure of $B_\rho(x) \cap \Omega$ is comparable to the measure of the ball $B_\rho(x)$. In other words,

$$(5) \qquad A\rho^n \le |B_\rho(x) \cap \Omega| \le \big(1 - A\big)\rho^n \quad \forall x \in \overline{\Omega},\ \forall \rho \in (0, \operatorname{diam}\Omega)$$

for some constant $A \in (0,1)$. This minimal regularity property excludes interior and exterior cusps at each point of $\partial\Omega$ and holds for example if $\partial\Omega$ has the uniform interior and exterior cone property. In particular, (5) is true in the cases of $C^1$-smooth, or Lipschitz, or Reifenberg flat boundaries (cf. [2]).

Let us finally recall that a function $u \in W_0^{1,m}(\Omega)$ is a *weak solution* to (1) if

$$\int_\Omega \boldsymbol{a}(x, u(x), Du(x)) \cdot D\kappa(x)\, dx + \int_\Omega b(x, u(x), Du(x))\kappa(x)\, dx = 0 \tag{6}$$

for each test function $\kappa \in W_0^{1,m}(\Omega)$. It should be noted that the convergence of the integrals above follows from (2) under the *sole* assumptions $\varphi \in L^{\frac{m}{m-1}}(\Omega)$ and $\psi \in L^{\frac{nm}{n(m-1)+m}}(\Omega)$ and these are the *minimal* ones to impose on the data of (1) in order that the concept of weak $W_0^{1,m}(\Omega)$ solution to (1) makes sense.

**2. Boundedness of the weak solutions.** The essential boundedness of the $W_0^{1,m}(\Omega)$-weak solutions to (1) is a classical fact due to LADYZHENSKAYA and URAL'TSEVA (see [3] and Theorem 7.1, Chapter IV in [1]) under the hypotheses (2) and (4) and when $\varphi$ and $\psi$ belong to $L^p$-spaces, that is,

$$\begin{cases} \varphi \in L^p(\Omega) \quad \text{with} \quad (m-1)p > n, \\ \psi \in L^q(\Omega) \quad \text{with} \quad mq > n. \end{cases} \tag{7}$$

Our aim is to extend that result to the *non-$L^p$ settings,* allowing $\varphi$ and $\psi$ to satisfy (3) instead of (7).

**Theorem 1.** *Under the assumptions* (2)–(5), *each $W_0^{1,m}(\Omega)$-weak solution of problem* (1) *is globally essentially bounded. Namely, there exists a constant $C$, depending on the data of* (1) *and on $\|Du\|_{L^m(\Omega)}$, such that*

$$\|u\|_{L^\infty(\Omega)} \le C. \tag{8}$$

To give an idea of the proof we note that $p > \dfrac{m}{m-1}$, $q > \dfrac{nm}{n(m-1)+m}$ and the reverse Hölder inequality (Gehring–Giaquinta–Modica Lemma, see Theorem 2.2, Chapter V in [4] or Chapter 6 in [5]) imply higher integrability of the gradient $Du$. Precisely, there is a number $r_0 > m$, close enough to $m$, such that $Du \in L^{r_0}(\Omega)$ with

$$\|Du\|_{L^{r_0}(\Omega)} \le C, \tag{9}$$

where $C$ depends on the data of (1) and on $\|Du\|_{L^m(\Omega)}$ in addition. If $n = m$ then (8) follows immediately from (9), $r_0 > m$ and the Sobolev theorem, so we concentrate on the case $n > m$.

Now, if $p$ and $q$ satisfy $(m-1)p > n$ and $mq > n$ (cf. (7)), then the claim (8) follows directly by the above cited result (Theorem 7.1, Chapter IV in [1]) of Ladyzhenskaya and Ural'tseva. Otherwise, we consider the measure

$$d\mathcal{M} := \left(\chi(x) + (\varphi(x))^{\frac{m}{m-1}} + \psi(x) + |u(x)|^{\frac{m^2}{n-m}}\right) dx, \tag{10}$$

where $\chi(x)$ is the indicator function of $\Omega$ and $dx$ stands for the Lebesgue measure.

An essential step in proving Theorem 1 is ensured by the following *trace inequality* due to ADAMS [6]:

**Lemma 1.** *Let $\mathcal{M}$ be a positive Radon measure with support in $\Omega$ and such that*

$$\mathcal{M}(B_\rho(x)) \leq K\rho^\alpha$$

*for each $x \in \mathbb{R}^n$ and each $\rho > 0$, where*

$$r\alpha = s(n-r), \quad 1 < r < s < \infty, \quad r < n.$$

*Then*

$$\left(\int_\Omega |v(x)|^s \, d\mathcal{M}\right)^{1/s} \leq CK^{1/s}\left(\int_\Omega |Dv(x)|^r \, dx\right)^{1/r} \quad \forall\, v \in W_0^{1,r}(\Omega)$$

*with $C = C(n,s,r)$.*

It is not difficult to check that, thanks to (3), (5) and (9), measure (10) satisfies

$$\mathcal{M}(B_\rho) \leq K\rho^{n-m+\varepsilon} \tag{11}$$

with $\varepsilon > 0$ and a constant $K$ depending on the data of (1) and on $\operatorname{diam}\Omega$.

Let now $k \geq 1$ be arbitrary and consider the function

$$v_k(x) := \max\{u(x)-k, 0\}$$

and the set

$$A_k := \big\{x \in \Omega\colon \; u(x) > k\big\}.$$

It is clear that $\operatorname{supp} v_k \subset \Omega_k$ and $v_k \in W_0^{1,m}(\Omega)$. Employing $v_k$ as a test function in (6) and using (2)–(4), (9), (11) and Lemma 1, we obtain that there exists a number $k_0 \geq 0$, such that

$$\int_{A_k} v_k(x)\, d\mathcal{M} \leq Ck\,(\mathcal{M}(A_k))^{1+\frac{\varepsilon}{m(n-m+\varepsilon)}} \quad \forall\, k \geq k_0.$$

With the setting

$$f(t) := \mathcal{M}(A_t),$$

the Cavalieri principle rewrites the last inequality into

$$\int_k^\infty f(t)\, dt \leq Ck\big(f(k)\big)^{1+\delta} \quad \forall\, k \geq k_0,\ \delta > 0. \tag{12}$$

At this point we invoke the following Hartman–Stampacchia maximum principle (see [7] and Lemma 5.1, Chapter II in [1]):

**Lemma 2.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-negative and non-increasing function. Suppose there exist constants $C > 0$, $k_0 \geq 0$, $\delta > 0$ and $\alpha \in [0, 1+\delta]$ such that*

$$\int_k^\infty f(t)\,dt \leq Ck^\alpha \big(f(k)\big)^{1+\delta} \quad \forall\ k \geq k_0.$$

*Then the function $f$ has the* finite time extinction property, *that is, there exists a number $k_{\max}$, depending on $C$, $k_0$, $\delta$, $\alpha$ and $\int_{k_0}^\infty f(t)\,dt$, such that*

$$f(k) = 0 \quad \forall\ k \geq k_{\max}.$$

It follows from (12) and Lemma 2 the existence of a number $k_{\max}$, depending on the data of (1) and on $\|Du\|_{L^m(\Omega)}$, such that

$$\mathcal{M}(A_k) = 0 \quad \forall\ k \geq k_{\max},$$

whence

$$u(x) \leq k_{\max} \quad \text{almost everywhere in } \Omega.$$

Repeating the same arguments to $-u(x)$ instead of $u(x)$, leads to an estimate from below for $u(x)$ which yields (8).

**Remark 1.** Comparing (3) with (7), it is clear that the decrease of the exponents $p$ and $q$ of Lebesgue integrability of the data $\varphi$ and $\psi$ is on account of the increase of the Morrey exponents $\lambda$ and $\mu$, and the conditions $(m-1)p+\lambda > n$ and $mq + \mu > n$ control the margins of these variations. In the particular case when $\lambda = \mu = 0$, our estimate (8) reduces to the classical one of Ladyzhenskaya and Ural'tseva from [1]. However, our Theorem 1 gives essential generalization to that of [1], because of the fact that taking $p < \dfrac{n}{m-1}$ and $q < \dfrac{n}{m}$, and bearing in mind the inclusions between the Morrey spaces (cf. [8]), there exist functions $\varphi \in L^{p,\lambda}(\Omega)$ with $(m-1)p + \lambda > n$ *not belonging to* $L^{\overline{p}}(\Omega)$ for all $\overline{p} \geq \dfrac{n}{m-1}$ and functions $\psi \in L^{q,\mu}(\Omega)$ with $mq + \mu > n$ *not belonging to* $L^{\overline{q}}(\Omega)$ for all $\overline{q} \geq \dfrac{n}{m}$.

Moreover, the requirements $\varphi \in L^{p,\lambda}(\Omega)$ with $(m-1)p + \lambda > n$ and $\psi \in L^{q,\mu}(\Omega)$ with $mq + \mu > n$, as asked in (3), are *sharp*, in order to have essential boundedness of the weak solutions to (1) even in the case of *linear equations.* This is shown by the next example (built on the counterexamples of Ladyzhenskaya and Ural'tseva [1]), where $m = 2$ for the sake of simplicity. Let $\Omega = \{x \in \mathbb{R}^n\colon\ |x| < R\}$ with $R < 1$. The *unbounded* function $u(x) = \log\left(\dfrac{\log|x|}{\log R}\right)$ is a $W_0^{1,2}(\Omega)$-weak solution to the equations

$$\Delta u = \psi(x) := \frac{n-2}{|x|^2 \log|x|} - \frac{1}{(|x|\log|x|)^2}$$

and

$$\Delta u = D_i\left(\varphi^i(x)\right), \quad \varphi^i(x) := \frac{x_i}{|x|^2 \log|x|}.$$

It is easy to check that $\varphi^i \in L^n(\Omega) \subset L^{\overline{p},n-\overline{p}}(\Omega)\ \forall\ \overline{p} \in (1,n]$, $\psi \in L^{\frac{n}{2}}(\Omega) \subset L^{\overline{q},n-2\overline{q}}(\Omega)\ \forall\ \overline{q} \in \left(1, \frac{n}{2}\right]$, but for each $\varepsilon > 0$ one has $\varphi^i \notin L^{\overline{p},n-\overline{p}+\varepsilon}(\Omega)$ and $\psi \notin L^{\overline{q},n-2\overline{q}+\varepsilon}(\Omega)$.

Employing suitable test function in (6), we get the following outgrowth of Theorem 1 when $n > m$.

**Corollary 1.** *Under the assumptions of Theorem* 1 *let* $n > m$. *Then the gradient of any* $W_0^{1,m}(\Omega)$*-weak solution to* (1) *belongs to the Morrey space* $L^{m,n-m}(\Omega)$.

**3. Hölder continuity of the weak solutions.** In paper [9] Serrin proved *local (interior)* Hölder continuity of the weak solutions to (1), assuming the following structure conditions on the data

$$\begin{cases} |\boldsymbol{a}(x,z,\xi)| \leq a|\xi|^{m-1} + b|z|^{m-1} + e \\ |b(x,z,\xi)| \leq c|\xi|^{m-1} + d|z|^{m-1} + f \\ \boldsymbol{a}(x,z,\xi)\cdot\xi \geq |\xi|^m - d|z|^m - g, \end{cases} \tag{13}$$

where $a = \text{const} > 0$, $m \in (1,n)$ and $b,\ldots,g$ are functions belonging to suitable $L^p$ spaces. The first to consider (1) with the structure (13) in the *non-$L^p$ settings* was Trudinger who in [10] generalized the Serrin results by allowing the functions $b,\ldots,g$ to belong to $L^{p,\varepsilon}$-spaces with *small enough* $\varepsilon > 0$. Finally, Zamboni proved in [11] *local Hölder continuity* of solutions to (1) under the structure (13) with coefficients taken in Morrey spaces.

Our second result implies global Hölder continuity of the weak solutions to (1).

**Theorem 2.** *Assume* (2)–(5). *Then each* $W_0^{1,m}(\Omega)$*-weak solution of problem* (1) *is globally Hölder continuous in* $\overline{\Omega}$ *with an exponent* $\alpha \in (0,1)$ *depending on the data of* (1) *and on* $\|Du\|_{L^m(\Omega)}$.

The proof makes use of the results of Theorem 1 and Corollary 1 and relies on the Moser iteration technique and Harnack inequalities.

It is worth noting that in the particular case of second-order semilinear elliptic equations the Hölder continuity follows *at once* by a classical linear result of Morrey (cf. [12]). Precisely, consider the Dirichlet problem

$$\begin{cases} u \in W_0^{1,2}(\Omega) \\ \operatorname{div}\left(a^{ij}(x,u)D_ju + a^i(x,u)\right) = b(x,u,Du) \quad \text{weakly in } \Omega \end{cases} \tag{14}$$

with Carathéodory nonlinear terms and where $\left\{a^{ij}(x,z)\right\}_{i,j=1}^n$ is a positively definite matrix of essentially bounded entries. Assume

• *Controlled growth conditions*: There exist a constant $\Lambda > 0$ and non-negative functions $\varphi \in L^{p,\lambda}(\Omega)$ with $p > 2$, $\lambda \in (0,n)$ and $p+\lambda > n$, and $\psi \in L^{q,\mu}(\Omega)$ with $q > 2n/(n+2)$, $\mu \in (0,n)$ and $2q+\mu > n$, such that

$$(15) \qquad \begin{cases} |a^i(x,z)| \leq \Lambda\Big(\varphi(x) + |z|^{\frac{n}{n-2}}\Big), \\ |b(x,z,\xi)| \leq \Lambda\Big(\psi(x) + |z|^{\frac{n+2}{n-2}} + |\xi|^{1+\frac{2}{n}}\Big) \end{cases}$$

for almost all $x \in \Omega$, all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

• *Uniform ellipticity*: $a^{ij} \in L^\infty(\Omega \times \mathbb{R})$ and there exists $\gamma > 0$ such that

$$(16) \qquad a^{ij}(x,z)\xi_i\xi_j \geq \gamma|\xi|^2$$

for almost all $x \in \Omega$, all $z \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

It is clear that (15) and (16) correspond to (2)–(3) and (4), respectively, in the settings of (14). Therefore, the essential boundedness of the $W_0^{1,2}(\Omega)$-weak solution to (14) follows from Theorem 1. Further on, using Corollary 1, (15) and the inclusion properties of the Morrey spaces (cf. [8]), it is not hard to see that

$$g^i(x) := a^i(x,u(x)) \in L^{p,\lambda}(\Omega), \quad p+\lambda > n,$$

and

$$f(x) := b(x,u(x),Du(x)) \in L^{q',\mu'}(\Omega)$$

with suitable $q'$ and $\mu'$ satisfying $2q'+\mu' > n$. Thus, setting $A^{ij}(x) := a^{ij}(x,u(x)) \in L^\infty(\Omega)$, we conclude that $u \in W_0^{1,2}(\Omega)$ is a weak solution to the *uniformly elliptic linear equation*

$$D_i\left(A^{ij}(x)D_ju(x)\right) = f(x) - D_i(g^i(x)).$$

This way, applying the results from [8] we get

**Theorem 3.** *Assume* (5), (15) *and* (16). *Then each* $W_0^{1,2}(\Omega)$*-weak solution of the semilinear Dirichlet problem* (14) *is globally Hölder continuous in* $\overline{\Omega}$ *with an exponent* $\alpha \in (0,1)$ *depending on the data of* (14) *and on* $\|Du\|_{L^2(\Omega)}$.

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