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CONDITION NUMBERS AND LOCAL PERTURBATION BOUNDS FOR THE MATRIX EQUATION $X^s \pm A^{\mathrm{H}} X^t A = Q$

Ivan P. Popchev, Vera A. Angelova

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Abstract

Norm-wise, mixed and component-wise condition numbers for the complex matrix equation $X^s \pm A^{\mathrm{H}} X^t A = Q$, with s, t real numbers, are derived. On the basis of the condition numbers, first order perturbation bounds are proposed as well.

Key words: perturbation analysis, norm-wise, mixed, component-wise condition numbers, local perturbation bound, Fréchet derivatives

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1. Introduction. The condition numbers are local (asymptotic) bounds for the perturbations in the solution as first order homogeneous functions of the perturbations in the data. They are valid for infinitesimal perturbations in the data and have relatively simple derivation and easy computation. Beside its independent use, the condition numbers are involved in the derivation of local perturbation bounds.

The measure of the sensitivity, in terms of the relative perturbations, is preferable for problems with data widely different in their magnitude. When the perturbations in the components of the data and/or the unknown matrix differ significantly, or/and if the input data are badly scaled or sparse, the component-wise perturbation analysis gives much tighter and revealing bounds than the norm-wise perturbation analysis.

In this paper, we apply the theory of norm-wise and component-wise perturbation analysis to derive norm-wise, mixed and component-wise condition numbers to the matrix equation

$$X^s \pm A^{\mathrm{H}} X^t A = Q, \qquad (1)$$

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where A is a nonsingular $n \times n$ complex matrix and Q is an Hermitian positive definite matrix. $A^{\rm H}$ stands for the conjugate transpose of A and both s and t are real numbers.

This paper is a sequel to [1], where the existence of solution and norm-wise perturbation analysis of equation (1) are considered. For norm-wise perturbation bounds to some particular cases of (1) see [2–7, 16] and the references therein. Comparison analysis of the effectiveness and the reliability of most of these bounds is given in [8, 9].

This paper is organized in 6 sections. In Section 2 definitions of the norm-wise, the mixed and the component-wise condition numbers are given. Section 3 is devoted to the first-order perturbation analysis of equation (1). Norm-wise absolute and relative condition numbers, as well as mixed and component-wise condition numbers of equation (1), are derived in Section 4. First order perturbation bounds for the solution of equation (1) are formulated in terms of the condition numbers proposed. In Section 5, the results are illustrated by a numerical example. Section 6 contains our conclusions.

2. Preliminary definitions and notations. We adopt the following notations: $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ are the sets of $n \times n$ complex and real matrices, respectively; $A^\top$ is the transpose of A; $A \otimes B = (a_{ij}B)$ is the Kronecker product of A and B; $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top$ is the vector representation of matrix A, where $A = [a_{ij}]$ and $a_1, a_2, \ldots, a_n \in \mathbb{C}^n$ are the columns of A; $\mathcal{P}_{n^2} \in \mathbb{R}^{n^2\times n^2}$ is the so-called vec-permutation matrix, such that $\mathrm{vec}(Y^\top) = \mathcal{P}_{n^2}\mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n\times n}$; $\Delta = \|\delta X\| + \|\delta A\| + \|\delta Q\|$; $x \preceq y$ is a partial order relation if $y - x \in \mathcal{K}$, $\mathcal{K}$ is a nonnegative cone; $\|.\|_2$ and $\|.\|_{\rm F}$ are the spectral and the Frobenius matrix norms, respectively, $\|.\|$ is a unitary invariant norm such as the spectral norm $\|\cdot\|_2$ or the Frobenius norm $\|\cdot\|_{\rm F}$. The notation ':=' stands for 'equal by definition'.

For our purposes, the following expressions of the norm-wise, the mixed and the component-wise condition numbers are used. For their definition and derivation, refer to [10].

Definition 1. *For a given problem $X = \Phi(B)$ with data $B := B_0 + \imath B_1 \in \mathbb{C}^{n\times n}$ ($\imath^2 = -1$), unknown matrix $X := X_0 + \imath X_1 \in \mathbb{C}^{n\times n}$ and perturbations in the data expressed by $\delta B := \delta B_0 + \imath\delta B_1 \in \mathbb{C}^{n\times n}$, the finite quantity*

$$K(B) := \lim_{\beta\to 0} \sup \left\{ \frac{\|\Phi(B + \delta B) - \Phi(B)\|}{\|\delta B)\|} : \delta B \neq 0, \|\delta B\| \leq \beta \right\} \tag{2}$$

is the absolute norm-wise condition number and the quantity

$$k(B) := K(B)\frac{\|B\|}{\|X\|} = K(B)\frac{\|B\|}{\|\Phi(B))\|} \tag{3}$$

is the relative norm-wise condition number of problem (Φ, B). On the basis of the norm-wise condition numbers local perturbation bounds for the solution X can be derived.

Denote by $x = \varphi(b)$ *the real vectorized embedding of the problem* $X = \Phi(B)$, *where* $\varphi := (\mathrm{vec} \circ \Phi)^{\mathcal{R}}$, $z := \mathrm{vec}(Z)^{\mathcal{R}} = \begin{bmatrix} \mathrm{vec}(Z_0) \\ \mathrm{vec}(Z_1) \end{bmatrix} \in \mathbb{R}^{2n^2}$ *for* $Z = Z_0 + \imath Z_1 \in \mathbb{C}^{n\times n}$, $Z_0, Z_1 \in \mathbb{R}^{n\times n}$ *and* Z *stands for* $X, B \in \mathbb{C}^{n\times n}$. *The realification of the vectorized transformation of the problem* (Φ, B) *is imposed to assure differentiability of* $(\mathrm{vec} \circ \Phi)$.

Definition 2. *For a given problem* $X = \Phi(B)$ *with vectorized embedding* $x = \varphi(b)$, *if the derivative* $\varphi'(b)$ *of* φ *at* b *exists, then the quantity*

$$
\begin{aligned}
(4) \qquad \hat{k}(b) \;&=\; \frac{\|\varphi'(b)\mathrm{diag}(b_1, \ldots, b_{2n^2})\|_\infty}{\|x\|_\infty} = \frac{\||\varphi'(b)|\,|b|\|_\infty}{\|x\|_\infty} \\
&\leq\; k(b) = \frac{\varphi'(b)\|_\infty \|b\|_\infty}{\|x\|_\infty}
\end{aligned}
$$

is the mixed relative condition number, where $k(b)$ *is the standard relative condition number* (3) *from Definition* 1, *taken with respect to the infinity norm* $\|.\|_\infty$. *If also the solution* x *has no zero components, the quantity*

$$
(5) \qquad \tilde{k}(b) := \|\mathrm{diag}(1/x_1, \ldots, 1/x_{2n^2})\varphi'(b)\mathrm{diag}(b_1, \ldots, b_{2n^2})\|_\infty
$$

is the relative component-wise condition number.

3. First order perturbation analysis. Denote by $S := (A,\ Q) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ the collection of matrix coefficients of equation (1). Let $\mathcal{S} \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$, such that for $S \in \mathcal{S}$ equation (1) has a solution $X \in \mathbb{C}_+^{n\times n}$. The existence of a positive definite solution to equation (1) is discussed in [1]. Thus, a certain subset of $\mathcal{S}$ is described.

Suppose that S varies over an open subset $\mathcal{S}^o$ of $\mathcal{S}$. In this case $X = \Phi(S)$, where the map $\Phi : \mathcal{S}^o \to \mathbb{C}_+^{n\times n}$ is Fréchet pseudo-differentiable [10].

Assume that the coefficients A, Q of equation (1) are slightly perturbed by δA, δQ, so as the perturbed equation

$$
\begin{aligned}
(6) \quad F(X + \delta X, S + \delta S) \;&:=\; (X + \delta X)^s \\
&\pm\; (A + \delta A)^{\mathrm{H}}(X + \delta X)^t(A + \delta A) - Q - \delta Q = 0
\end{aligned}
$$

has solution $X + \delta X$ in the neighbourhood of the unperturbed solution X. Subtracting the unperturbed equation (1) from the perturbed equation (6) and dropping the second and higher-order terms, we get

$$
(7) \qquad F_X(X, S)(\delta X) \;\approx\; -F_Q(X, S)(\delta Q) - F_A(X, S)(\delta A).
$$

Here $F_Q(X, S)(Z) := -Z$ is the partial Fréchet derivative of (6) in Q, calculated at the point (X, S). The term $F_X(X, S)(Z) := \mathcal{F}(s, X)(Z) \pm A^{\mathrm{H}}\mathcal{F}(t, X)(Z)A$ is the partial Fréchet derivative of (6) in X at the point (X, S), where $\mathcal{F}(p, X)(Z)$, $p = s, t$ denotes the Fréchet derivative of the function $X \to X^p$, $p \in \mathbb{R}$ at the point

X: $(X+Y)^p = \mathcal{F}(p,X)(Y) + \mathrm{O}(\|Y\|^2)$, $Y \to 0$. Expressions of the derivatives $\mathcal{F}(p,X)$ and of their matrices $L_p := \mathrm{Mat}\{\mathcal{F}(p,X)\}$ for different value of $p \in \mathbb{R}$: $p = \pm r$, $p = \pm 1/l$, $p = \pm r/l$, $r, l = 1, 2, \ldots$, are listed on Table 1. The term $F_A(X,S)(Z) := \pm A^{\mathrm{H}} X^t Z \pm Z^{\mathrm{H}} X^t A$ is the Fréchet pseudo derivative of (6) in A. The operator $F_A(X,S)$ is additive but not homogeneous. This implies to apply the technique of additive operators [10].

Table 1

Expressions of the derivatives $\mathcal{F}(p,X)$ and their matrices L_p [11, 12]

p	$\mathcal{F}(p,X)(Y)$	L_p
r	$\sum_{k=0}^{r-1} X^{r-1-k} Y X^k$	$\sum_{k=0}^{r-1} (X^k)^\top \otimes X^{r-1-k}$
$-r$	$-\sum_{k=0}^{r-1} X^{-1-k} Y X^{k-r}$	$-\sum_{k=0}^{r-1} (X^{k-r})^\top \otimes X^{-(1+k)}$
$1/l$	$\mathcal{F}^{-1}(l, X^{1/l})(Y)$	$\left(\sum_{k=0}^{l-1} (X^{k/l})^\top \otimes X^{(l-1-k)/l}\right)^{-1}$
$-1/l$	$\mathcal{F}^{-1}(-l, X^{-1/l})(Y)$	$\left(-\sum_{k=0}^{l-1} (X^{(k-l)/l})^\top \otimes X^{-(1+k)/l}\right)^{-1}$
r/l	$\mathcal{F}(r, X^{1/l}) \circ \mathcal{F}^{-1}(l, x^{1/l})(Y)$	$\left(\sum_{k=0}^{r-1} (X^{k/l})^\top \otimes X^{(r-1-k)/l}\right) L_{1/l}$
$-r/l$	$\mathcal{F}(-1, X^{r/l}) \circ \mathcal{F}(r/l, X)(Y)$	$-\left((X^{-r/l})^\top \otimes X^{-r/l}\right) L_{r/l}$

Suppose that the operator $F_X(X,S)$ is invertible, i.e. its matrix $L := \mathrm{Mat}\{F_X(X,S)\} = L_s \pm (A^\top \otimes A^{\mathrm{H}}) L_t$ is non-singular. For the perturbation δX in the solution X we obtain (7)

$$\delta X \approx -F_X^{-1} \circ F_Q(\delta Q) - F_X^{-1} \circ F_A(\delta A),$$

or in a vector form

$$\mathrm{vec}(\delta X) \approx W_Q \mathrm{vec}(\delta Q) + (W_A + W_{\bar{A}}) \mathrm{vec}(\delta A). \tag{8}$$

Here $W_Z := -L^{-1} L_Z = W_{Z0} + \imath W_{Z1} \in \mathbb{C}^{n^2 \times n^2}$, $\imath^2 = -1$, where L_Z are the matrices of the operators $F_Z(X,S)$ for $Z = Q, A, \bar{A}$: $L_Q := \mathrm{Mat}\{F_Q(X,S)\} = -I_{n^2}$, $L_A := \mathrm{Mat}\{F_A(X,S)\} = I \otimes A^{\mathrm{H}} X^t$ and $L_{\bar{A}} := \mathrm{Mat}\{F_{\bar{A}}(X,S)\} = ((X^t A)^\top \otimes I)\mathcal{P}_{n^2}$.

Applying the technique [10] for additive complex operators in expression (8), we obtain its realification

$$\mathrm{vec}(\delta X)^{\mathcal{R}} \approx W_Q^{\mathcal{R}} \mathrm{vec}(\delta Q)^{\mathcal{R}} + \Theta_A \mathrm{vec}(\delta A)^{\mathcal{R}}, \tag{9}$$

or

$$v_X \approx \Theta \begin{bmatrix} v_Q \\ v_A \end{bmatrix}, \tag{10}$$

where

$$\begin{aligned}
v_Z &:= \operatorname{vec}(\delta Z)^{\mathcal{R}} = \begin{bmatrix} \operatorname{vec}(\delta Z_0) \\ \operatorname{vec}(\delta Z_1) \end{bmatrix} \in \mathbb{R}^{2n^2}, \\
& \qquad\qquad\qquad \text{for } \delta Z = \delta Z_0 + \imath \delta Z_1 \text{ and } Z = X, Q, A, \\
\Theta &:= \begin{bmatrix} \Theta_Q & \Theta_A \end{bmatrix} \in \mathbb{R}^{2n^2 \times 4n^2}, \\
\Theta_Q &:= W_Q^{\mathcal{R}} = \begin{bmatrix} W_{Q0} & -W_{Q1} \\ W_{Q1} & W_{Q0} \end{bmatrix} \in \mathbb{R}^{2n^2 \times 2n^2}, \\
\Theta_A &:= \Theta_A(W_A, W_{\bar{A}}) = \begin{bmatrix} W_{A0} + W_{\bar{A}0} & W_{\bar{A}1} - W_{A1} \\ W_{A1} + W_{\bar{A}1} & W_{A0} - W_{\bar{A}0} \end{bmatrix} \in \mathbb{R}^{2n^2 \times 2n^2}.
\end{aligned} \tag{11}$$

4. Condition numbers. ***4.1. Norm-wise condition numbers.*** Denote $z := \|Z\|_{\mathrm{F}}$, $\delta_Z := \|\delta Z\|_{\mathrm{F}}/z$, for $Z = A, Q$ and $\delta_X := \|\delta X\|_{\mathrm{F}}/\|X\|_{\mathrm{F}}$.

Theorem 1. *For the solution X of equation (1), with data matrices satisfying $(A, Q) \in \mathcal{S}^0$, the following norm-wise absolute and relative condition numbers are valid:*

absolute condition numbers

$$K_Z := \|\Theta_Z\|_2, \quad Z = A, Q, \tag{12}$$

relative condition numbers

$$k_z := \frac{K_Z \|Z\|_2}{\|X\|_2}, \quad Z = A, Q, \tag{13}$$

$$k_1 := \lim_{\varepsilon \to 0} \sup \left\{ \frac{\delta_X}{\varepsilon} \; : \; \sqrt{\delta_Q^2 + \delta_A^2} \le \varepsilon \right\} = \frac{\| \begin{bmatrix} q\Theta_Q & a\Theta_A \end{bmatrix} \|_2}{\|X\|_{\mathrm{F}}}. \tag{14}$$

Proof. Expressions (12) and (13) for the absolute and the relative norm-wise condition numbers follow directly from definitions (2), (3) and expression (9).

For the proof of k_1 (14) rewrite expression (10) as

$$v_X \approx \begin{bmatrix} q\Theta_Q & a\Theta_A \end{bmatrix} \begin{bmatrix} v_Q/q \\ v_A/a \end{bmatrix} \tag{15}$$

and take the spectral norm of both sides of (15). One obtains

$$\begin{aligned}
\delta_X &\le \frac{1}{\|X\|_{\mathrm{F}}} \| \begin{bmatrix} q\Theta_Q & a\Theta_A \end{bmatrix} \|_2 \left\| \begin{bmatrix} v_Q/q \\ v_A/a \end{bmatrix} \right\|_2 \\
&= \frac{1}{\|X\|_{\mathrm{F}}} \left\| \begin{bmatrix} q\Theta_Q & a\Theta_A \end{bmatrix} \right\|_2 \sqrt{\delta_A^2 + \delta_Q^2}.
\end{aligned}$$

Hence, (14) holds. □

As function $F(X, S)$ is Lipschitzian at S, the heuristic rule [13] may be applied to estimate the actual error in the computed solution for computing environment with rounding unit *macheps*: when *macheps* $k(S) < 1$, then about $-\log_{10}(\mathit{macheps}\, k(S))$ are the true decimal digits in the result.

The local estimate $\mathrm{est}(\delta)$ from [1] in terms of the norm-wise condition numbers of Theorem 1 is

$$\delta_X \le \mathrm{est}(\delta) := \min\{\mathrm{est}_2(\delta), \mathrm{est}_3(\delta)\}, \tag{16}$$

where $\delta := \begin{bmatrix} \delta_Q & \delta_A \end{bmatrix} \in \mathbb{R}^2_+$ is the vector of relative data perturbations in Frobenius norm and

$$\begin{aligned}
\mathrm{est}_2(\delta) &:= k_1\sqrt{\delta_Q^2 + \delta_A^2},\\
\mathrm{est}_3(\delta) &:= \sqrt{k_Q^2\delta_Q^2 + k_A^2\delta_A^2 + \frac{2l}{\|X\|_{\mathrm{F}}^2}},\\
l &:= q^2\|\Theta_Q^\top\Theta_Q\|_2\delta_Q^2 + aq(\|\Theta_Q^\top\Theta_A\|_2 + \|\Theta_A^\top\Theta_Q\|_2)\delta_Q\delta_A + a^2\|\Theta_A^\top\Theta_A\|_2\delta_A^2.
\end{aligned}$$

4.2. Mixed and component-wise condition numbers. Denote $\sigma := \begin{bmatrix} \mathrm{vec}(\delta Q)^{\mathcal{R}} \\ \mathrm{vec}(\delta A)^{\mathcal{R}} \end{bmatrix} = [\sigma_{ij}]$, $i, j = 1, 2, \ldots, 2n^2$. The concepts of the component-wise perturbation analysis are developed in [10, 14, 15].

Theorem 2. *Let $S \in \mathcal{S}^0$.*

1. *Let $x \neq 0$. According to Definition 2, the mixed condition number (4) for equation (1) is*

$$\hat{k}(\sigma) := \frac{\||\Theta|\,|\sigma|\|_\infty}{\|x\|_\infty}. \tag{17}$$

2. *If $x_j \neq 0$, for $j = 1, 2, \ldots, 2n^2$, then the component-wise condition number (18) of equation (1), according to Definition 2 is*

$$\tilde{k}(\sigma) := \|(|\Theta|\,|\sigma|)\,./|x|\|_\infty\,, \tag{18}$$

where Θ is defined in (11).

Proof. The mixed condition number (17) and the component-wise condition number (18) follow directly from expression (9), (11). □

Let $|\delta A_0| \preceq \varepsilon_0|A_0|$, $|\delta A_1| \preceq \varepsilon_1|A_1|$, ..., $|\delta A_k| \preceq \varepsilon_k|A_k|$ and $\varepsilon := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$. Then for equation (1) the following mixed and component-wise perturbation bounds, based on the mixed and the component-wise condition numbers from Theorem 2, are valid

$$\|\mathrm{vec}(\delta X)\|_\infty/\|x\|_\infty \le \varepsilon\,\hat{k}(\sigma) + \mathrm{O}(\varepsilon),\ \varepsilon \to 0, \tag{19}$$

$$\||\mathrm{vec}(\delta X)|./|x|\|_\infty \le \varepsilon\,\tilde{k}(\sigma) + \mathrm{O}(\varepsilon),\ \varepsilon \to 0. \tag{20}$$

The component-wise perturbation bound is an estimate of the sensitivity of the elements of the solution to perturbations in the elements of the data. Its use is convenient when the elements of the data vary in a special way, e.g., when some of them remain constant.

5. Numerical example. Consider equation $X^s - A^\top X^t A = Q$, with $s = 1$, $t = 3/4$. Let $A = \frac{d}{\|A_0\|} A_0$, where $d = \frac{19}{20} - 10^{-2}$ and $A_0 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$;

$X = \begin{bmatrix} 2.5 & 1 & 1 & 1 & 1 \\ 1 & 2.5 & 1 & 1 & 1 \\ 1 & 1 & 2.5 & 1 & 1 \\ 1 & 1 & 1 & 2.5 & 1 \\ 1 & 1 & 1 & 1 & 2.5 \end{bmatrix}$ and $Q = X^s - A^\top X^t A$. Consider the perturbed matrix equation $(X + \delta X) - (A + \delta A)^\top (X + \delta X)^{3/4}(A + \delta A) = Q + \delta Q$, where $\delta A = 10^{-j} \frac{C^* + C}{\|C^* + C\|}$ for $j = 4, 6, 8, 10$, C is a random matrix generated by the function `randn` of MATLAB,

$$\delta X = 10^{-j} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$\delta Q = (X + \delta X) + (A + \delta A)^\top (X + \delta X)^{3/4}(A + \delta A) - Q.$$

This example is taken from [2]. On Table 2 are given the results obtained for the relative norm-wise condition numbers k_A and k_Q (13), the overall relative norm-wise condition number k_1 (14), the mixed condition number $\hat{k}(\sigma)$ (17), the component-wise condition number $\tilde{k}(\sigma)$ (18) and the ratio $\frac{\mathrm{est}(\delta)}{\delta_x}$ of the local estimate $\mathrm{est}(\delta)$ (16), based on the norm-wise condition numbers to the estimated

Table 2

Numerical example

Bound	$j = 4$	$j = 6$	$j = 8$	$j = 10$
k_A	0.9649	0.9649	0.9649	0.9649
k_Q	0.8731	0.8731	0.8731	0.8731
$\hat{k}(\sigma)$	4.0301	4.0301	4.0299	4.2729
$\tilde{k}(\sigma)$	4.0301	4.0301	4.0299	4.7428
k_1	1.0701	1.0700	1.0700	1.0700
$\mathrm{est}(\delta)/\delta_x$	1.6308	1.6306	1.6306	1.6306

value–the relative perturbations δ_X in the solution X. The numerical example demonstrates the effectiveness of the bounds proposed.

6. Conclusions. In this paper, explicit expressions for the norm-wise, mixed and component-wise condition numbers of the complex matrix equation (1) are obtained. First order bounds for the perturbations in the computed solution are proposed, as well. The condition numbers and the local perturbation bounds proposed allow easy computable and fast estimate of the accuracy of the computed solution. The effectiveness of the bounds proposed is demonstrated with a numerical example.

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Institute of Information and Communication Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria
e-mail: ipopchev@iit.bas.bg, vangelova@iit.bas.bg