PARACOMPACTNESS, SELECTIONS AND FACTORIZATION PRINCIPLES

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Abstract

Factorization principles for set-valued mappings are obtained. These principles are applied in theorems for selections of set-valued mappings from paracompact spaces.

Key words: selection, paracompact space, factorization

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1. Introduction. All considered spaces are assumed to be $T_0$-spaces. Our terminology comes, as a rule, from [13,17,21]. In the present paper we continue the investigations begun in [9-12].

A topological space $X$ is called paracompact if $X$ is Hausdorff and every open cover of $X$ has a locally finite open refinement.

The cardinal number $l(X) = \min\{m : \text{every open cover of } X \text{ has an open refinement of cardinality } \leq m\}$ is the Lindelöf number of $X$.

Let $X$ and $Y$ be non-empty topological spaces. A set-valued mapping $\theta : X \to Y$ assigns to every $x \in X$ a non-empty subset $\theta(x)$ of $Y$. If $\phi, \psi : X \to Y$ are set-valued mappings and $\phi(x) \subseteq \psi(x)$ for every $x \in X$, then $\phi$ is called a selection of $\psi$.

Let $\theta : X \to Y$ be a set-valued mapping and let $A \subseteq X$ and $B \subseteq Y$. The set $\theta^{-1}(B) = \{x \in X : \theta(x) \cap B \neq \emptyset\}$ is the inverse image of the set $B$, $\theta(A) = \bigcup\{\theta(x) : x \in A\}$ is the image of the set $A$ and $\theta^{n+1}(A) = \theta(\theta^{n}(A))$ is the $n+1$-image of the set $A$. The set $\theta^\infty(A) = \bigcup\{\theta^n(A) : n \in \mathbb{N}\}$ is the largest image of the set $A$.

A set-valued mapping $\theta : X \to Y$ is called lower (upper) semi-continuous if for every open (closed) subset $H$ of $Y$ the set $\theta^{-1}(H)$ is open (closed) in $X$.

A set-valued mapping $\theta : X \to Y$ is called perfect if it is an upper semicontinuous compact-valued mapping, $\theta^{-1}(y)$ is compact for every $y \in Y$ and $\theta : X \to \theta(X)$ is a closed mapping.
Let $X$ and $Y$ be topological spaces and $\theta : X \to Y$ be a set-valued mapping. A weak $r$-factorization (weak right-factorization) for $\theta$ is a triple $(Z, g, \phi)$, where $g : Z \to Y$ is a continuous single-valued mapping from the space $Z$ into $Y$ and $\phi : X \to Z$ is a set-valued mapping such that $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$. We conserve the word factorization (right-factorization) for $\theta$ when $g(\phi(x)) = \theta(x)$ for every $x \in X$. If $\phi$ is a single-valued continuous mapping (respectively, lower or upper semi-continuous), then the weak $r$-factorization $(Z, g, \phi)$ is called single-valued (respectively, lower or upper semi-continuous) weak $r$-factorization. Note that every weak $r$-factorization $(Z, g, \phi)$ valued (respectively, lower or upper semi-continuous), then the weak $r$-factorization $(Z, g, \phi)$ is a lower semi-continuous mapping $\phi$.

The set-valued factorizations (weak factorizations) from $^{19,20}$, one may name it left-factorizations (respectively weak left-factorizations). Recall that a factorization (weak factorization) for $\theta$ is a triple $(Z, g, \phi)$, where $g : X \to Z$ is a continuous single-valued mapping from the space $X$ into $Z$ and $\phi : Z \to Y$ is a set-valued mapping such that $\phi(g(x)) = \theta(x)$ (respectively $\phi(g(x)) \subseteq \theta(x)$) for every $x \in X$ (see $^{19,20}$).

The concept of factorization of single-valued mappings was introduced by S. Mardešić in $^{16}$. The case of set-valued mappings was examined in $^{19,20}$.

Our notion of the weak $r$-factorization is distinct from the weak factorization. Really, let $\theta : X \to Y$ be a single-valued continuous mapping. If $(Z, g, \phi)$ is a weak factorization for $\theta$, then $\phi$ is a single-valued mapping. In this case we have the Mardešić's $^{16}$ factorization. Let $f : Z \to Y$ be a single-valued mapping of the space $Z$ onto the space $Y$. We put $\psi(x) = f^{-1}(\theta(x))$ for any $x \in X$. Then $(Z, f, \psi)$ is a weak $r$-factorization for $\theta$. Obviously that $(Z, f, \psi)$ is a weak factorization for $\theta$ if and only if the mapping $f$ is one-to-one.

The notion of $r$-factorization implicitly was used in $^{9}$ and $^{7}$.

For every finite-dimensional (infinite-dimensional) normal space $X$ there exists a single-valued continuous mapping $\theta : X \to Y$ into a finite-dimensional (infinite-dimensional) separable metric space $Y$ such that for every single-valued weak $r$-factorization $(Z, g, \phi)$ it follows that $\dim Z \geq \dim X$. Really, let $\dim X \geq n$. Then there exists a finite open cover $\gamma = \{U_1, U_2, \ldots, U_m\}$ such that for any open refinement $\xi$ of $\gamma$ we have $\ord(\xi) = \max\{\vert A \vert : A \subseteq \{1, 2, \ldots, m\}, \cap \{U_i : i \in A \} \neq \emptyset\} \geq n + 1$. Fix the continuous functions $\{f_i : X \to I = [0, 1] : i \leq m\}$ such that $X \setminus U_i \subseteq f_i^{-1}(0)$ for every $i \leq m$ and $\Sigma\{f_i(x) : i \leq m\} = 1$ for every $x \in X$. Then $\theta = \theta_n : X \to Y_n = \theta_n(X) \subseteq I^m$, where $\theta_n(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for each $x \in X$, is the desired mapping for the case $\dim X \geq n$. If $\dim X = \infty$, then the mapping $\theta$ is the diagonal product of the mappings $\theta_n$. This fact is not true for set-valued lower (neither for upper) semi-continuous weak $r$-factorizations.

For set-valued mappings there exist some similar results. Denote by $\ord(\phi) = \sup\{\tau : \tau < \vert \phi(x)\vert \}$ it for some $x \in X$ the order of the set-valued mapping $\phi : X \to Y$. For every normal space $X$ and any natural number $n \leq \dim X$ there exist a lower semi-continuous mapping $\varphi : X \to Y$ and an upper semi-continuous mapping $\psi : X \to Y$ into a finite discrete space $Y$ such that $\psi$ is a selection for...
φ and for every lower or upper semi-continuous factorization \((Z, g, \phi)\) of \(φ\) it follows \(n \leq \text{ord}(φ)\).

If \(θ : X → Y\) is an upper semi-continuous compact-valued mapping of a regular space \(X\) into a regular space \(Y\), then \(Gr(θ)\) is closed in \(X × Y\) and there exists a closed subset \(Z ⊆ Gr(θ)\) such that the projection \(g : Z → X\) is a perfect irreducible mapping (see [13]). The mapping \(g\) is a homeomorphism if and only if \(X\) is extremely disconnected. In this case \(Z\) is the graph of a single-valued continuous selection of \(θ\) (see [15]).

2. Factorization principles. A set-valued mapping with a property \(Q\) is called a \(Q\)-mapping. A property \(Q\) of set-valued mappings is called a perfect property if for every set-valued \(Q\)-mapping \(θ : X → Y\) from a space \(X\) into a metrizable space \(Y\), every perfect mapping \(h : Z → Y\) from a metrizable space \(Z\) onto \(Y\) and every subspace \(H\) of \(Z\), such that \(h(H) = Y\), the mapping \(φ : X → Z\), where \(φ(x) = cl_Z(H \cap h^{-1}(θ(x)))\) for every \(x \in X\), is a \(Q\)-mapping.

Let \(τ\) be an infinite cardinal number. A mapping \(θ : X → Y\) is called \(τ\)-dense if \(l(θ(θ^{-1}(H))) ≤ τ\) for every \(H ⊆ Y\) such that \(l(H) ≤ τ\).

The property of a set-valued mapping to be \(τ\)-dense is a perfect property.

There is a very long list of classes of topological spaces characterized in terms of special selections of lower set-valued mappings into completely metrizable spaces (see for example [3–5, 7, 9–12, 14, 17–19, 21]).

The following principles are useful tools for the investigation of selection problems.

Theorems 1 (2) U-factorization principle (L-factorization principle). For a space \(X\) and a perfect property \(Q\) of set-valued mappings the following assertions are equivalent:

1. For every lower semi-continuous mapping \(θ : X → Y\) into a completely metrizable \(Y\), there exist a completely metrizable space \(Z\), a continuous single-valued mapping \(g : Z → Y\) and an upper semi-continuous mapping \(ψ : X → Z\) with the property \(Q\) such that \(g(ψ(x)) ⊆ \theta(x)\) for every \(x \in X\) (a lower semi-continuous mapping \(φ : X → Z\) with the property \(Q\) such that \(g(φ(x)) ⊆ \theta(x)\) for every \(x \in X\)).

2. For every lower semi-continuous mapping \(θ : X → Y\) into a completely metrizable \(Y\) there exist a completely metrizable zero-dimensional space \(Z\), a continuous single-valued mapping \(g : Z → Y\) and an upper semi-continuous mapping \(ψ : X → Z\) with the property \(Q\) such that \(g(ψ(x)) ⊆ \theta(x)\) for every \(x \in X\) (a lower semi-continuous mapping \(φ : X → Z\) with the property \(Q\) such that \(g(φ(x)) ⊆ \theta(x)\) for every \(x \in X\)).

Theorem 3 (M-factorization principle). For a space \(X\) and a perfect property \(Q\) of set-valued mappings the following assertions are equivalent:

1. For every lower semi-continuous mapping \(θ : X → Y\) into a completely metrizable \(Y\) there exist a completely metrizable space \(Z\), a continuous single-valued mapping \(g : Z → Y\), a lower semi-continuous mapping \(φ : X → Z\) and
an upper semi-continuous mapping $\psi : X \to Z$ both $\phi$ and $\psi$ with the property $Q$

such that $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

2. For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ there exist a completely metrizable zero-dimensional space $Z$, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous mapping $\phi : X \to Z$ and an upper semi-continuous mapping $\psi : X \to Z$ both with the property $Q$ such that $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

**Proof of the three factorization principles.** Let $\phi_1, \psi_1 : X \to Z_1$ be two mappings with property $Q$ from $X$ into a completely metrizable space $Z_1$ and $h : Z_1 \to Y$ be a single-valued continuous mapping. There exist a completely metrizable zero-dimensional space $Z$, a subset $H$ of $Z$, a perfect mapping $f : Z \to Z_1$ such that $f(H) = Z_1$ and $f \upharpoonright H : H \to Z_1$ is an open mapping (see \cite{2,8}). Put $\psi(x) = f^{-1}(\psi_1(x))$ and $\phi(x) = \text{cl}_Z(H \cap f^{-1}(\phi_1(x)))$ for $x \in X$ and $g(z) = h(f(z))$ for every $z \in Z$. Then $g : Z \to Y$ is a single-valued continuous mapping and $\phi, \psi : X \to Z$ are set-valued mappings with the property $Q$. If $\psi_1$ is upper semi-continuous, then $\psi$ is upper semi-continuous, too. If $\phi_1$ is lower semi-continuous, then $\phi$ is lower semi-continuous, too. This proves the implication $1 \Rightarrow 2$ of the factorization principles. The implication $2 \Rightarrow 1$ is obvious for any factorization principle. □

The construction from the proof above may be used in the proof of the following:

**Theorem 4.** For a $T_0$-space $X$ the following assertions are equivalent:

1. $X$ is a paracompact space.

2. For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a completely metrizable $Y$ there exist an upper semi-continuous mapping $\psi : X \to Y$ such that $\psi(x) \subseteq \theta(x)$ for every $x \in X$.

3. For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a completely metrizable $Y$ there exist a completely metrizable zero-dimensional space $Z$, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued mapping $\psi : X \to Z$ such that $\phi(x) \subseteq \psi(x)$, $\psi^\infty(x)$ is a separable space and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

**Proof.** From the factorization theorems from \cite{19,20} it follows that there exist a complete metric space $Z_1$, a continuous single-valued mapping $f : X \to Z_1$ and a lower semi-continuous compact-valued mapping $\lambda : Z_1 \to Y$ such that $\lambda(f(x)) \subseteq \theta(x)$ for any $x \in X$.

There exist a completely metrizable zero-dimensional space $Z$, a subset $H$ of $Z$ and a perfect mapping $h : Z \to Z_1$ such that $h(H) = Z_1$ and $h \upharpoonright H : H \to Z_1$ is an open compact mapping (see \cite{2,8}). For the lower semi-continuous compact-valued mapping $\theta_1 : Z \to Y$ where $\theta_1(z) = \lambda(h(z))$ for every $z \in Z$, there exists a single-valued continuous selection $g : Z \to Y$. Put $\psi(x) = h^{-1}(f(x))$ and $\varphi(x) = H \cap \psi(x)$ for every $x \in X$. By construction $\psi^\infty(x) = \psi(x)$ is compact. □
3. On paracompact $p$-spaces. A space $X$ is called a paracompact $p$-space if it is Hausdorff and there exists a perfect single-valued mapping $\theta : X \to Y$ onto some metrizable space $Y$ (see [1]).

Theorem 5. For a $T_0$-space $X$ the following assertions are equivalent:
1. $X$ is a paracompact $p$-space.
2. For every lower semi-continuous mapping $\theta : X \to Y$ there exist a completely metrizable space $Z$, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and a perfect set-valued mapping $\psi : X \to Z$ such that $\phi$ and $\psi$ are $\tau$-dense mappings for every infinite cardinal number $\tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.
3. For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ there exist a completely metrizable zero-dimensional space $Z$, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and a perfect set-valued mapping $\psi : X \to Z$ such that $\phi$ and $\psi$ are $\tau$-dense mappings for every infinite cardinal number $\tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

Proof. Implications $2 \Rightarrow 3 \Rightarrow 2$ follow from the M-factorizations principles. Implication $2 \Rightarrow 1$ is obvious. Let $\theta : X \to Y$ be a lower semi-continuous mapping from a paracompact $p$-space $X$ into a completely metrizable space $Y$. By virtue of Michael’s selection theorem [18], there exist a lower semi-continuous compact-valued mapping $\phi_1 : X \to Y$ and an upper semi-continuous compact-valued mapping $\psi_1 : X \to Y$ such that $\phi_1(x) \subseteq \psi_1(x) \subseteq \theta(x)$ for every $x \in X$. There exist a completely metrizable space $S$, a subspace $S_1$ of $S$ and a perfect single-valued mapping $f : X \to S_1$ from $X$ onto $S_1$. Let $\phi(x) = \phi_1(x) \times \{f(x)\}$ and $\psi(x) = \psi_1(x) \times \{f(x)\}$ for every $x \in X$. Then $\phi : X \to Y \times S$ is lower semi-continuous and $\psi : X \to Y \times S$ is a perfect set-valued mapping. Consider the projection $g : Y \times S \to Y$. Then $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) = \psi_1(x) \subseteq \theta(x)$ for every $x \in X$.

4. On strongly paracompact spaces. Let $\mathcal{P}$ be a property of topological spaces. The space $X$ has property $\text{loc} \mathcal{P}$ if for any point $x \in X$ there exist an open subset $U$ of $X$ and a subset $H$ of $X$ such that $x \in U \subseteq H \subseteq \text{cl}_X U$ and $H$ has the property $\mathcal{P}$. In particular, $\text{loc}(X) \leq \tau$ if there exists an open cover $\gamma$ of $X$ such that $l(U) \leq \tau$ for any $U \in \gamma$.

Theorem 6. For a $T_0$-space $X$ the following are equivalent:
1. $X$ is strongly paracompact.
2. For every infinite cardinal number $\tau$ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ with $\text{loc}(Y) \leq \tau$ there exist a completely metrizable space $Z$, an open continuous single-valued mapping $g : Z \to Y$ onto $Y$, a lower semi-continuous $\tau$-dense mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued $\tau$-dense mapping $\psi : X \to Z$ such that $\text{loc}(Z) \leq \tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$. 

3. For every infinite cardinal number $\tau$ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ with $\text{loc}(Y) \leq \tau$ there exist a zero-dimensional completely metrizable space $Z$, an open continuous single-valued mapping $g : Z \to Y$ onto $Y$, a lower semi-continuous $\tau$-dense mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued $\tau$-dense mapping $\psi : X \to Z$ such that $\text{loc}(Z) \leq \tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

4. The space $X$ is regular and for every infinite cardinal number $\tau$ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ with $\text{loc}(Y) \leq \tau$ there exist a completely metrizable space $Z$, an open continuous single-valued mapping $g : Z \to Y$ onto $Y$, a lower semi-continuous $\tau$-dense mapping $\phi : X \to Z$ such that $\text{loc}(Z) \leq \tau$ and $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$.

5. For every infinite cardinal number $\tau$ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable $Y$ with $\text{loc}(Y) \leq \tau$ there exist a completely metrizable space $Z$, an open continuous single-valued mapping $g : Z \to Y$ onto $Y$ an upper semi-continuous $\tau$-dense mapping $\psi : X \to Z$ such that $\text{loc}(Z) \leq \tau$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

**Proof.** Implications 3 $\Rightarrow$ 4 and 3 $\Rightarrow$ 5 are obvious. Implication 2 $\Rightarrow$ 3 follows from $M$-factorization principle.

Let $X$ be a strongly paracompact space, $\theta : X \to Y$ be a lower semi-continuous mapping into a complete metrizable, $\tau$ be an infinite cardinal number and $\text{loc}(Y) \leq \tau$. Put $\gamma = \{V : V$ is open in $Y$ and $l(V) \leq \tau\}$. Obviously $Y = \bigcup \gamma$. There exists an open star-finite cover $\xi = \{U_\alpha : \alpha \in A\}$ of the space $X$ such that for every $\alpha \in A$ there exists $V_\alpha \in \gamma$ with $U_\alpha \subseteq \theta^{-1}(V_\alpha)$. One can assume that $Y = \bigcup\{V_\alpha : \alpha \in A\}$. Denote by $Z$ the discrete sum $\bigoplus\{V_\alpha \times \{\alpha\} : \alpha \in A\}$. Put $g(y, \alpha) = y$ for every $y \in V_\alpha$ and $\alpha \in A$. Then $g : Z \to Y$ is an open continuous single-valued mapping and the space $Z$ is completely metrizable.

For every $x \in X$ put $\theta_1(x) = \bigcup\{V_\alpha \cap \theta(x) \times \{\alpha\} : x \in U_\alpha, \alpha \in A\}$. Then $\theta_1 : X \to Z$ is a lower semi-continuous mapping and $g(\theta_1(x)) \subseteq \theta(x)$ for every $x \in X$. By construction, $\text{loc}(Z) \leq \tau$.

By virtue of Michael’s selection theorem [18], there exist a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued mapping $\psi : X \to Z$ such that $\phi(x) \subseteq \psi(x) \subseteq \theta_1(x)$ for every $x \in X$. There exists a decomposition $\{A_\beta : \beta \in B\}$ of $A$ such that $A = \bigcup\{A_\beta : \beta \in B\}$, $A_\beta$ is finite or countable for every $\beta \in B$ and if $\beta, \mu \in B$ and $X_\beta = \bigcup\{U_\alpha : \alpha \in A_\beta\}$ then $X_\beta = X_\mu$, or $X_\beta \cap X_\mu = \emptyset$.

Let $\beta \in B$. Put $Z_\beta = \bigcup\{V_\alpha \times \{\alpha\} : \alpha \in A_\beta\}$. Then $l(Z_\beta) \leq \tau$, $X_\beta = \phi^{-1}(Z_\beta) = \psi^{-1}(Z_\beta)$ and $\phi(X_\beta) \subseteq \psi(X_\beta) \subseteq Z_\beta$. Hence $\phi^\infty(x) \subseteq \psi^\infty(x) \subseteq Z_\beta$ for every $x \in X_\beta$. It follows that $l(\phi^\infty(x)) \leq \tau$ and $l(\psi^\infty(x)) \leq \tau$ for every $x \in X$.

Fix $H \subseteq Z$ with $l(H) \leq \tau$. Then $B(H) = \{\beta \in B : H \cap Z_\beta \neq \emptyset\}$ is a set of cardinality less then $\tau$. Hence $l\bigcup\{Z_\beta : \beta \in B(H)\} \leq \tau$ and $\psi^{-1}(H) \subseteq \bigcup\{Z_\beta : \beta \in B(H)\}$. Therefore $l(\psi^{-1}(H)) \leq \tau$. Implication 1 $\Rightarrow$ 2 is proved.

Suppose that $\xi = \{U_\alpha : \alpha \in Y\}$ is an open cover of $X$. Endow $Y$ with the
Consider a completely metrizable space $Z$, a continuous single-valued mapping $g : Z \to Y$ and an $\omega$-dense mapping $\varphi : X \to Z$ such that $g(\varphi(x)) \subseteq \theta(x)$ for every $x \in X$. The space $Z$ is locally separable. Thus every subspace of $Z$ is strongly paracompact. There exists a star-finite open cover $\gamma = \{W_\mu : \mu \in M\}$ such that for every $\mu \in M$ the space $W_\mu$ is separable and there exists $y(\mu) \in Y$ with $W_\mu \subseteq g^{-1}(y(\mu))$. Then $\varphi^{-1}(\text{cl}_ZW_\mu) \subseteq U_{y(\mu)}$ for every $\mu \in M$.

Let $V_\mu = \varphi^{-1}(W_\mu)$ and $H_\mu = \varphi^{-1}(\text{cl}_ZW_\mu)$. Since $\varphi$ is $\tau$-dense, the space $\varphi^\infty(H_\mu)$ is separable for every $\mu \in M$. In particular, $\{H_\mu : \mu \in M\}$ is a star-countable refinement of $\xi$.

Assume that the mapping $\varphi$ is upper semi-continuous. Then $\xi' = \{H_\mu : \mu \in M\}$ is a conservative star-countable closed refinement of the cover $\xi$. Thus the space $X$ is strongly paracompact (see [9]). Implication 5 $\Rightarrow$ 1 is proved.

Assume that $\varphi$ is lower semi-continuous and $X$ is regular. Then $\{V_\mu : \mu \in M\}$ is an open star-countable refinement of $\xi$. Thus $X$ is strongly paracompact. Implication 4 $\Rightarrow$ 1 and the theorem are proved. $\square$

**Remark 7.** A set-valued mapping $\phi : X \to Z$ into a discrete space $Z$ is $\omega$-dense if and only if the set $\phi^2(x)$ is countable for every $x \in X$. Thus the characterization of strong paracompactness in [9] can be regarded as a corollary of Theorem 6.

**Remark 8.** The recent article [14] contains the uniform characterization of strong paracompactness. Theorem 6 is a topological characterization of strong paracompactness. Theorems 4 and 5 show that the following conditions are essential: $\text{loc}(Z) \leq \text{loc}(Y)$; the selection $\varphi$ is $\text{loc}(Y)$-dense.

The method in the proof of Theorem 6 uses essentially the $\tau$-density of the factorization.

**Example 9.** Let $I = [0, 1]$ and $I_t = [0, 1] \times \{t\}$ for every $t \in I$. Denote by $Y$ the discrete sum $\bigoplus\{I_t : t \in I\} \bigoplus I$. Denote by $X$ the space which underlying set is $Y$ with the discrete topology on $\bigoplus\{I_t : t \in I\}$ and with the natural topology on the open-and-closed subspace $I$. Consider the mapping $\theta : X \to Y$ where $\theta(t) = \{t\}$ for $t \in I$, $\theta((x,t)) = \{(x,t)\}$ for $x \in I$ and $t > 0$ and $\theta((t,0)) = \{t, (t,0)\}$ for every $t \in I$. Then the mapping $\theta$ is continuous (i.e. lower and upper semi-continuous), compact-valued, $\phi^2(x) = \phi^\infty(x)$ and $|\phi^\infty(x)| \leq 2$ for every $x \in X$. The family $\gamma = \{I, I_t : t \in I\}$ is a discrete cover of $Y$ and $\theta^{-1}(\gamma)$ is a locally finite open cover of $X$. But $\theta^{-1}(\gamma)$ is not star-finite. The set $I$ is separable and the set $\theta(\theta^{-1}(I))$ is not separable (in both spaces $X$ and $Y$). The mapping $\theta$ has a single-valued selection. $\square$

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