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PARACOMPACTNESS, SELECTIONS AND FACTORIZATION PRINCIPLES

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Abstract

Factorization principles for set-valued mappings are obtained. These principles are applied in theorems for selections of set-valued mappings from paracompact spaces.
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1. Introduction. All considered spaces are assumed to be T_0-spaces. Our terminology comes, as a rule, from [13, 17, 21]. In the present paper we continue the investigations begun in [9–12].

A topological space X is called *paracompact* if X is Hausdorff and every open cover of X has a locally finite open refinement.

The cardinal number $l(X) = \min\{m : \textit{every open cover of } X \textit{ has an open refinement of cardinality} \leq m\}$ is the Lindelöf number of X.

Let X and Y be non-empty topological spaces. A *set-valued mapping* $\theta : X \to Y$ assigns to every $x \in X$ a non-empty subset $\theta(x)$ of Y. If $\phi,\ \psi : X \to Y$ are set-valued mappings and $\phi(x) \subseteq \psi(x)$ for every $x \in X$, then ϕ is called *a selection* of ψ.

Let $\theta : X \to Y$ be a set-valued mapping and let $A \subseteq X$ and $B \subseteq Y$. The set $\theta^{-1}(B) = \{x \in X : \theta(x) \bigcap B \neq \varnothing\}$ is *the inverse image* of the set B, $\theta(A) = \theta^1(A) = \bigcup\{\theta(x) : x \in A\}$ is *the image* of the set A and $\theta^{n+1}(A) = \theta(\theta^{-1}(\theta^n(A)))$ is *the* $n+1$*-image* of the set A. The set $\theta^\infty(A) = \bigcup\{\theta^n(A) : n \in \mathbb{N}\}$ is *the largest image* of the set A.

A set-valued mapping $\theta : X \to Y$ is called *lower (upper) semi-continuous* if for every open (closed) subset H of Y the set $\theta^{-1}(H)$ is open (closed) in X.

A set-valued mapping $\theta : X \to Y$ is called perfect if it is an upper semi-continuous compact-valued mapping, $\theta^{-1}(y)$ is compact for every $y \in Y$ and $\theta : X \to \theta(X)$ is a closed mapping.

Let X and Y be topological spaces and $\theta : X \to Y$ be a set-valued mapping. A weak r-factorization (weak right-factorization) for θ is a triple (Z, g, ϕ), where $g : Z \to Y$ is a continuous single-valued mapping from the space Z into Y and $\phi : X \to Z$ is a set-valued mapping such that $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$. We conserve the word r-factorization (right-factorization) for θ when $g(\phi(x)) = \theta(x)$ for every $x \in X$. If ϕ is a single-valued continuous mapping (respectively, lower or upper semi-continuous), then the weak r-factorization (Z, g, ϕ) is called single-valued (respectively, lower or upper semi-continuous) weak r-factorization. Note that every weak r-factorization (Z, g, ϕ) generates a selection $g \circ \phi : X \to Y$ for θ. The set-valued factorizations (weak factorizations) from [19, 20], one may name it left-factorizations (respectively weak left-factorizations). Recall that a factorization (weak factorization) for θ is a triple (Z, g, ϕ), where $g : X \to Z$ is a continuous single-valued mapping from the space X into Z and $\phi : Z \to Y$ is a set-valued mapping such that $\phi(g(x)) = \theta(x)$ (respectively $\phi(g(x)) \subseteq \theta(x)$) for every $x \in X$ (see [19, 20]).

The concept of factorization of single-valued mappings was introduced by S. Mardešić in [16]. The case of set-valued mappings was examined in [19, 20].

Our notion of the weak r-factorization is distinct from the weak factorization. Really, let $\theta : X \longrightarrow Y$ be a single-valued continuous mapping. If (Z, g, ϕ) is a weak factorization for θ, then ϕ is a single-valued mapping. In this case we have the Mardešić's [16] factorization. Let $f : Z \longrightarrow Y$ be a single-valued mapping of the space Z onto the space Y. We put $\psi(x) = f^{-1}(\theta(x))$ for any $x \in X$. Then (Z, f, ψ) is a weak r-factorization for θ. Obviously that (Z, f, ψ) is a weak factorization for θ if and only if the mapping f is one-to-one.

The notion of r-factorization implicitly was used in [6] and [7].

For every finite-dimensional (infinite-dimensional) normal space X there exists a single-valued continuous mapping $\theta : X \to Y$ into a finite-dimensional (infinite-dimensional) separable metric space Y such that for every single-valued weak r-factorization $(Z,\ g,\ \phi)$ it follows that $\dim Z \geq \dim X$. Really, let $\dim X \geq n$. Then there exists a finite open cover $\gamma = \{U_1, U_2, \ldots, U_m\}$ such that for any open refinement ξ of γ we have $\mathrm{ord}\,(\xi) = \max\{|A| : A \subseteq \{1, 2, \ldots, m\},\ \cap\{U_i : i \in A\} \neq \varnothing\} \geq n + 1$. Fix the continuous functions $\{f_i : X \to I = [0, 1] : i \leq m\}$ such that $X \setminus U_i \subseteq f_i^{-1}(0)$ for every $i \leq m$ and $\Sigma\{f_i(x) : i \leq m\} = 1$ for every $x \in X$. Then $\theta = \theta_n : X \longrightarrow Y_n = \theta_n(X) \subseteq I^m$, where $\theta_n(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for each $x \in X$, is the desired mapping for the case $\dim X \geq n$. If $\dim X = \infty$, then the mapping θ is the diagonal product of the mappings θ_n. This fact is not true for set-valued lower (neither for upper) semi-continuous weak r-factorizations.

For set-valued mappings there exist some similar results. Denote by $\mathrm{ord}\,(\phi) = \sup\{\tau : \tau < |\phi(x)|$ it for some $x \in X\}$ the order of the set-valued mapping $\phi : X \to Y$. For every normal space X and any natural number $n \leq \dim X$ there exist a lower semi-continuous mapping $\varphi : X \to Y$ and an upper semi-continuous mapping $\psi : X \to Y$ into a finite discrete space Y such that ψ is a selection for

φ and for every lower or upper semi-continuous factorization $(Z,\ g,\ \phi)$ of φ it follows $n \leq \mathrm{ord}\,(\phi)$.

If $\theta : X \to Y$ is an upper semi-continuous compact-valued mapping of a regular space X into a regular space Y, then $Gr(\theta)$ is closed in $X \times Y$ and there exists a closed subset $Z \subseteq Gr(\theta)$ such that the projection $g : Z \to X$ is a perfect irreducible mapping (see [13]). The mapping g is a homeomorphism if and only if X is extremely disconnected. In this case Z is the graph of a single-valued continuous selection of θ (see [15]).

2. Factorization principles. A set-valued mapping with a property Q is called a Q-mapping. A property Q of set-valued mappings is called a perfect property if for every set-valued Q-mapping $\theta : X \to Y$ from a space X into a metrizable space Y, every perfect mapping $h : Z \to Y$ from a metrizable space Z onto Y and every subspace H of Z, such that $h(H) = Y$, the mapping $\phi : X \to Z$, where $\phi(x) = cl_Z(H \cap h^{-1}(\theta(x)))$ for every $x \in X$, is a Q-mapping.

Let τ be an infinite cardinal number. A mapping $\theta : X \to Y$ is called τ-dense if $l(\theta(\theta^{-1}(H))) \leq \tau$ for every $H \subseteq Y$ such that $l(H) \leq \tau$.

The property of a set-valued mapping to be τ-dense is a perfect property.

There is a very long list of classes of topological spaces characterized in terms of special selections of lower set-valued mappings into completely metrizable spaces (see for example [3–5, 7, 9–12, 14, 17–19, 21]).

The following principles are useful tools for the investigation of selection problems.

Theorems 1 (2) U-factorization principle (L-factorization principle). *For a space X and a perfect property Q of set-valued mappings the following assertions are equivalent*:

1. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y, there exist a completely metrizable space Z, a continuous single-valued mapping $g : Z \to Y$ and an upper semi-continuous mapping $\psi : X \to Z$ with the property Q such that $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$ (a lower semi-continuous mapping $\phi : X \to Z$ with the property Q such that $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$).*

2. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable zero-dimensional space Z, a continuous single-valued mapping $g : Z \to Y$ and an upper semi-continuous mapping $\psi : X \to Z$ with the property Q such that $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$ (a lower semi-continuous mapping $\phi : X \to Z$ with the property Q such that $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$).*

Theorem 3 (M-factorization principle). *For a space X and a perfect property Q of set-valued mappings the following assertions are equivalent*:

1. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable space Z, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous mapping $\phi : X \to Z$ and*

an upper semi-continuous mapping $\psi : X \to Z$ both ϕ and ψ with the property Q such that $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.

2. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable zero-dimensional space Z, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous mapping $\phi : X \to Z$ and an upper semi-continuous mapping $\psi : X \to Z$ both with the property Q such that $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

Proof of the three factorization principles. Let $\phi_1, \psi_1 : X \to Z_1$ be two mappings with property Q from X into a completely metrizable space Z_1 and $h : Z_1 \to Y$ be a single-valued continuous mapping. There exist a completely metrizable zero-dimensional space Z, a subset H of Z, a perfect mapping $f : Z \to Z_1$ such that $f(H) = Z_1$ and $f \restriction H : H \to Z_1$ is an open mapping (see [2,8]). Put $\psi(x) = f^{-1}(\psi_1(x))$ and $\phi(x) = \operatorname{cl}_Z(H \cap f^{-1}(\phi_1(x)))$ for $x \in X$ and $g(z) = h(f(z))$ for every $z \in Z$. Then $g : Z \to Y$ is a single-valued continuous mapping and $\phi, \psi : X \to Z$ are set-valued mappings with the property Q. If ψ_1 is upper semi-continuous, then ψ is upper semi-continuous, too. If ϕ_1 is lower semi-continuous, then ϕ is lower semi-continuous, too. This proves the implication $1 \Rightarrow 2$ of the factorization principles. The implication $2 \Rightarrow 1$ is obvious for any factorization principle. □

The construction from the proof above may be used in the proof of the following:

Theorem 4. *For a T_0-space X the following assertions are equivalent:*

1. *X is a paracompact space.*

2. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a completely metrizable Y there exist an upper semi-continuous mapping $\psi : X \to Y$ such that $\psi(x) \subseteq \theta(x)$ for every $x \in X$.*

3. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable zero-dimensional space Z, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued mapping $\psi : X \to Z$ such that $\phi(x) \subseteq \psi(x)$, $\psi^\infty(x)$ is a separable space and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

Proof. From the factorization theorems from [19,20] it follows that there exist a complete metric space Z_1, a continuous single-valued mapping $f : X \to Z_1$ and a lower semi-continuous compact-valued mapping $\lambda : Z_1 \to Y$ such that $\lambda(f(x)) \subseteq \theta(x)$ for any $x \in X$.

There exist a completely metrizable zero-dimensional space Z, a subset H of Z and a perfect mapping $h : Z \to Z_1$ such that $h(H) = Z_1$ and $h \restriction H : H \to Z_1$ is an open compact mapping (see [2,8]). For the lower semi-continuous compact-valued mapping $\theta_1 : Z \to Y$ where $\theta_1(z) = \lambda(h(z))$ for every $z \in Z$, there exists a single-valued continuous selection $g : Z \to Y$. Put $\psi(x) = h^{-1}(f(x))$ and $\varphi(x) = H \cap \psi(x)$ for every $x \in X$. By construction $\psi^\infty(x) = \psi(x)$ is compact. □

3. On paracompact *p*-spaces. A space X is called a paracompact p-space if it is Hausdorff and there exists a perfect single-valued mapping $\theta : X \to Y$ onto some metrizable space Y (see [1]).

Theorem 5. *For a T_0-space X the following assertions are equivalent*:

1. *X is a paracompact p-space.*

2. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable space Z, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and a perfect set-valued mapping $\psi : X \to Z$ such that ϕ and ψ are τ-dense mappings for every infinite cardinal number τ, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

3. *For every lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y there exist a completely metrizable zero-dimensional space Z, a continuous single-valued mapping $g : Z \to Y$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and a perfect set-valued mapping $\psi : X \to Z$ such that ϕ and ψ are τ-dense mappings for every infinite cardinal number τ, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

Proof. Implications $2 \Rightarrow 3 \Rightarrow 2$ follow from the M-factorizations principles. Implication $2 \Rightarrow 1$ is obvious. Let $\theta : X \to Y$ be a lower semi-continuous mapping from a paracompact p-space X into a completely metrizable space Y. By virtue of Michael's selection theorem [18], there exist a lower semi-continuous compact-valued mapping $\phi_1 : X \to Y$ and an upper semi-continuous compact-valued mapping $\psi_1 : X \to Y$ such that $\phi_1(x) \subseteq \psi_1(x) \subseteq \theta(x)$ for every $x \in X$. There exist a completely metrizable space S, a subspace S_1 of S and a perfect single-valued mapping $f : X \to S_1$ from X onto S_1. Let $\phi(x) = \phi_1(x) \times \{f(x)\}$ and $\psi(x) = \psi_1(x) \times \{f(x)\}$ for every $x \in X$. Then $\phi : X \to Y \times S$ is lower semi-continuous and $\psi : X \to Y \times S$ is a perfect set-valued mapping. Consider the projection $g : Y \times S \to Y$. Then $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) = \psi_1(x) \subseteq \theta(x)$ for every $x \in X$. □

4. On strongly paracompact spaces. Let $\mathcal{P}$ be a property of topological spaces. The space X has property $\mathrm{loc}\,\mathcal{P}$ if for any point $x \in X$ there exist an open subset U of X and a subset H of X such that $x \in U \subseteq H \subseteq \mathrm{cl}_X U$ and H has the property $\mathcal{P}$. In particular, $\mathrm{locl}\,(X) \leq \tau$ if there exists an open cover γ of X such that $l(U) \leq \tau$ for any $U \in \gamma$.

Theorem 6. *For a T_0-space X the following are equivalent*:

1. *X is strongly paracompact.*

2. *For every infinite cardinal number τ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y with $locl(Y) \leq \tau$ there exist a completely metrizable space Z, an open continuous single-valued mapping $g : Z \to Y$ onto Y, a lower semi-continuous τ-dense mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued τ-dense mapping $\psi : X \to Z$ such that $locl(Z) \leq \tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

3. *For every infinite cardinal number τ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y with locl$(Y) \leq \tau$ there exist a zero-dimensional completely metrizable space Z, an open continuous single-valued mapping $g : Z \to Y$ onto Y, a lower semi-continuous τ-dense mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued τ-dense mapping $\psi : X \to Z$ such that locl$(Z) \leq \tau$, $\phi(x) \subseteq \psi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

4. *The space X is regular and for every infinite cardinal number τ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y with locl$(Y) \leq \tau$ there exist a completely metrizable space Z, an open continuous single-valued mapping $g : Z \to Y$ onto Y, a lower semi-continuous τ-dense mapping $\phi : X \to Z$ such that locl$(Z) \leq \tau$ and $g(\phi(x)) \subseteq \theta(x)$ for every $x \in X$.*

5. *For every infinite cardinal number τ and a lower semi-continuous mapping $\theta : X \to Y$ into a completely metrizable Y with locl$(Y) \leq \tau$ there exist a completely metrizable space Z, an open continuous single-valued mapping $g : Z \to Y$ onto Y an upper semi-continuous τ-dense mapping $\psi : X \to Z$ such that locl$(Z) \leq \tau$ and $g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$.*

Proof. Implications $3 \Rightarrow 4$ and $3 \Rightarrow 5$ are obvious. Implication $2 \Rightarrow 3$ follows from M-factorization principle.

Let X be a strongly paracompact space, $\theta : X \to Y$ be a lower semi-continuous mapping into a complete metrizable, τ be an infinite cardinal number and $\mathrm{locl}\,(Y) \leq \tau$. Put $\gamma = \{V : V$ is open in Y and $l(V) \leq \tau\}$. Obviously $Y = \bigcup \gamma$. There exists an open star-finite cover $\xi = \{U_\alpha : \alpha \in A\}$ of the space X such that for every $\alpha \in A$ there exists $V_\alpha \in \gamma$ with $U_\alpha \subseteq \theta^{-1}(V_\alpha)$. One can assume that $Y = \bigcup\{V_\alpha : \alpha \in A\}$. Denote by Z the discrete sum $\oplus\{V_\alpha \times \{\alpha\} : \alpha \in A\}$. Put $g(y, \alpha) = y$ for every $y \in V_\alpha$ and $\alpha \in A$. Then $g : Z \to Y$ is an open continuous single-valued mapping and the space Z is completely metrizable.

For every $x \in X$ put $\theta_1(x) = \cup\{(V_\alpha \cap \theta(x)) \times \{\alpha\} : x \in U_\alpha, \alpha \in A\}$. Then $\theta_1 : X \to Z$ is a lower semi-continuous mapping and $g(\theta_1(x)) \subseteq \theta(x)$ for every $x \in X$. By construction, $\mathrm{locl}\,(Z) \leq \tau$.

By virtue of Michael's selection theorem [18], there exist a lower semi-continuous compact-valued mapping $\phi : X \to Z$ and an upper semi-continuous compact-valued mapping $\psi : X \to Z$ such that $\phi(x) \subseteq \psi(x) \subseteq \theta_1(x)$ for every $x \in X$. There exists a decomposition $\{A_\beta : \beta \in B\}$ of A such that $A = \bigcup\{A_\beta : \beta \in B\}$, A_β is finite or countable for every $\beta \in B$ and if $\beta, \mu \in B$ and $X_\beta = \bigcup\{U_\alpha : \alpha \in A_\beta\}$ then $X_\beta = X_\mu$, or $X_\beta \cap X_\mu = \varnothing$.

Let $\beta \in B$. Put $Z_\beta = \bigcup\{V_\alpha \times \{\alpha\} : \alpha \in A_\beta\}$. Then $l(Z_\beta) \leq \tau$, $X_\beta = \phi^{-1}(Z_\beta) = \psi^{-1}(Z_\beta)$ and $\phi(X_\beta) \subseteq \psi(X_\beta) \subseteq Z_\beta$. Hence $\phi^\infty(x) \subseteq \psi^\infty(x) \subseteq Z_\beta$ for every $x \in X_\beta$. It follows that $l(\phi^\infty(x)) \leq \tau$ and $l(\psi^\infty(x)) \leq \tau$ for every $x \in X$.

Fix $H \subseteq Z$ with $l(H) \leq \tau$. Then $B(H) = \{\beta \in B : H \cap Z_\beta \neq \varnothing\}$ is a set of cardinality less then τ. Hence $l(\bigcup\{Z_\beta : \beta \in B(H)\}) \leq \tau$ and $\psi(\psi^{-1}(H)) \subseteq \bigcup\{Z_\beta : \beta \in B(H)\}$. Therefore $l(\psi(\psi^{-1}(H))) \leq \tau$. Implication $1 \Rightarrow 2$ is proved.

Suppose that $\xi = \{U_\alpha : \alpha \in Y\}$ is an open cover of X. Endow Y with the discrete topology. Then $\mathrm{locl}\,(Y) = \omega$ where ω is the countable cardinal. The mapping $\theta : X \to Y$, where $\theta(x) = \{y \in Y : x \in U_y\}$ for every $x \in X$ is lower semi-continuous.

Consider a completely metrizable space Z, a continuous single-valued mapping $g : Z \to Y$ and an ω-dense mapping $\varphi : X \to Z$ such that $g(\varphi(x)) \subseteq \theta(x)$ for every $x \in X$. The space Z is locally separable. Thus every subspace of Z is strongly paracompact. There exists a star-finite open cover $\gamma = \{W_\mu : \mu \in M\}$ such that for every $\mu \in M$ the space W_μ is separable and there exists $y(\mu) \in Y$ with $W_\mu \subseteq g^{-1}(y(\mu))$. Then $\varphi^{-1}(\mathrm{cl}\,_Z W_\mu) \subseteq U_{y(\mu)}$ for every $\mu \in M$.

Let $V_\mu = \varphi^{-1}(W_\mu)$ and $H_\mu = \varphi^{-1}(\mathrm{cl}\,_Z W_\mu)$. Since φ is τ-dense, the space $\varphi^\infty(H_\mu)$ is separable for every $\mu \in M$. In particular, $\{H_\mu : \mu \in M\}$ is a star-countable refinement of ξ.

Assume that the mapping φ is upper semi-continuous. Then $\xi' = \{H_\mu : \mu \in M\}$ is a conservative star-countable closed refinement of the cover ξ. Thus the space X is strongly paracompact (see [9]). Implication $5 \Rightarrow 1$ is proved.

Assume that φ is lower semi-continuous and X is regular. Then $\{V_\mu : \mu \in M\}$ is an open star-countable refinement of ξ. Thus X is strongly paracompact. Implication $4 \Rightarrow 1$ and the theorem are proved. □

Remark 7. *A set-valued mapping $\phi : X \to Z$ into a discrete space Z is ω-dense if and only if the set $\phi^2(x)$ is countable for every $x \in X$. Thus the characterization of strong paracompactness in [9] can be regarded as a corollary of Theorem 6.*

Remark 8. *The recent article [14] contains the uniform characterization of strong paracompactness. Theorem* 6 *is a topological characterization of strong paracompactness. Theorems* 4 *and* 5 *show that the following conditions are essential*: $\mathrm{locl}\,(Z) \le locl(Y)$; *the selection φ is* $\mathrm{locl}\,(Y)$*-dense.*

The method in the proof of Theorem 6 uses essentially the τ-density of the factorization.

Example 9. *Let $I = [0,1]$ and $I_t = [0,1] \times \{t\}$ for every $t \in I$. Denote by Y the discrete sum $\oplus\{I_t : t \in I\} \oplus I$. Denote by X the space which underlying set is Y with the discrete topology on $\oplus\{I_t : t \in I\}$ and with the natural topology on the open-and-closed subspace I. Consider the mapping $\theta : X \to Y$ where $\theta(t) = \{t\}$ for $t \in I$, $\theta((x,t)) = \{(x,t)\}$ for $x \in I$ and $t > 0$ and $\theta((t,0)) = \{t,(t,0)\}$ for every $t \in I$. Then the mapping θ is continuous* (*i.e. lower and upper semi-continuous*), *compact-valued, $\phi^2(x) = \phi^\infty(x)$ and $\mid \phi^\infty(x) \mid \le 2$ for every $x \in X$. The family $\gamma = \{I, I_t : t \in I\}$ is a discrete cover of Y and $\theta^{-1}(\gamma)$ is a locally finite open cover of X. But $\theta^{-1}(\gamma)$ is not star-finite. The set I is separable and the set $\theta(\theta^{-1}(I))$ is not separable* (*in both spaces X and Y*). *The mapping θ has a single-valued selection.* □

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