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ON THE NONLINEAR SDOF FREE VIBRATION ANALYSIS

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Abstract

The paper summarizes and discusses the approaches to the nonlinear SDOF free vibration behaviour to be analyzed and included in force vibration models. In brief, the general structure of the equation of motion, given in a Fourier like series form, is demonstrated. Bi-linear and multi-linear methods are compared with the so-called method of the harmonic balance from theoretical point of view. Such a method can be applied in the case of preliminary known, or predicted, total mass displacements. By a numerical example the results obtained through the discussed methods are verified by a FEM model.

Key words: nonlinear SDOF vibration, the harmonic balance method, nonlinear FEM application

1. Introduction. The subject of the paper is a non-standard approach to the eigenvalues and eigenvectors of non-linear elastic systems to be evaluated. Such a basis can be used for detailed harmonic, periodic, transient or spectral analysis [1–6]. For application of the method the ultimate displacements of the system must be known in advance.

The general form of the equation of motion for nonlinear SDOF free vibrating system with neglecting the damping can be written as

$$m\ddot{u}\,(x) + f_s\,(x) = 0, \tag{1}$$

where $m\ddot{u}\,(x)$ denotes the inertial force due to velocity change and $f_s\,(x)$ is the force, caused by the stiffness properties of the system. Here the relation $(f_s\,(x) - u\,(x))$ is assumed to be in some general (nonlinear) form, but the produced strains in the system remain elastic.

In such cases, the nonlinear SDOF free vibration is not harmonical but always the motion is periodical. An easy way to be expressed is based on the Fourier like series form written as

$$u\,(x) = A_0 + A_1 \cos\tilde{\omega}\,t + B_1 \sin\tilde{\omega}\,t + A_2 \cos 2\tilde{\omega}\,t + B_2 \sin 2\tilde{\omega}\,t + \cdots \tag{2}$$

or

$$u\,(x) = A_0 + \sum_{i=1}^{\infty} (A_i \cos i\tilde{\omega}\,t + B_i \sin i\tilde{\omega}\,t). \tag{3}$$

The coefficients A_i and B_i are called harmonic coefficients, and the coefficient A_0 is related to the shift of the centre of vibration, if such exist. For practical use, the infinite series, given by eq. (2) or eq. (3), can be limited to an appropriate chosen number, say n. Then, for the calculation of the free vibration position of the mass point, a finite number of terms can be used.

2. Multi-linear method. Bi-linear or multi-linear methods are based on bi-linear or multi-linear $(f_s\,(x) - u\,(x))$ relations, respectively, illustrated in Fig. 1. The equation of motion (1) acquires the forms:

For bi-linear relation

$$m\,\ddot{u}\;(x) + k_1 u\;(x) \pm k_2\,[u\;(x) - u_1] = 0. \tag{4}$$

For multi-linear relation

$$m\;\ddot{u}\;(x) + k_1 u\;(x) \pm \sum_{i=2}^{m} k_{ix}\;[u\;(x) - u_{i-1}] = 0. \tag{5}$$

In the above equations $\pm k_1$ is the ratio $f_s\,(x)\,/u\,(x)$ when $u\,(x) \in [0; u_1]$, k_2 is the change of the same relation for the second interval of the graph, k_i is the change for the i-th interval, by a comparison with the $(i-1)$-th interval, see Fig. 2. The coefficients u_{i-1} are the initial values of $u\,(x)$ for the i-th interval, and the sign $\pm$ denotes increasing or decreasing of the stiffness. The form, given by eq. (5), is proposed in [7].

In the structural mechanics problems, usually decreasing of the stiffness is used.

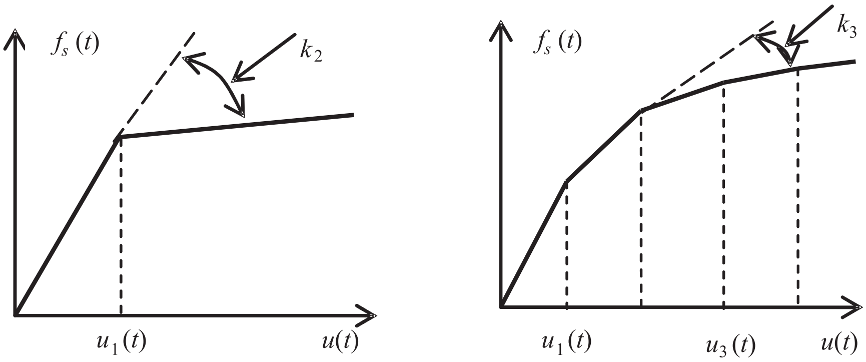

Fig. 1. Bi-linear or multi-linear $f_s(x) - u(x)$ representation

The solution of eq. (4) can be written as

$$u\ (x) = \frac{k_2}{k_1 \pm k_2} u_1 (1 - \cos\hat{\omega}\, t) + A \cos\hat{\omega}\, t, \tag{6}$$

where $\hat{\omega} = \sqrt{(k_1 \pm k_2)/m}$. Time t_1 related to u_1 can be calculated by

$$t_1 = \frac{1}{\hat{\omega}} \arccos \frac{k_1}{(k_1 \pm k_2) A - k_2 u_1} u_1. \tag{7}$$

The multi-linear method gives good approximation when the relation

$$(f_s(x) - u(x))$$

can be given in a multi-line form.

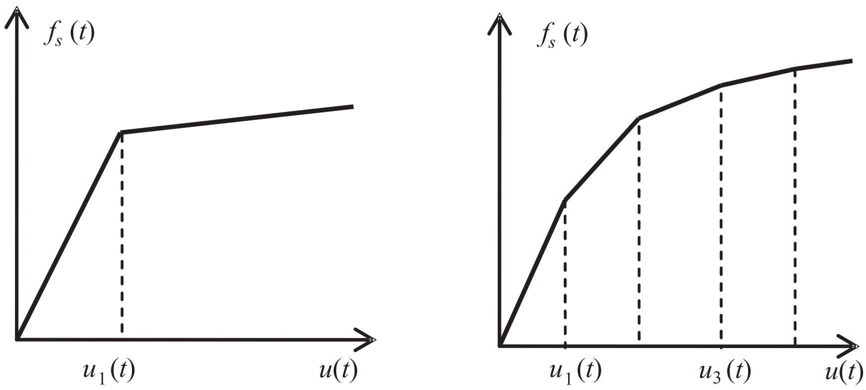

Fig. 2. Illustrations of the k_i coefficients

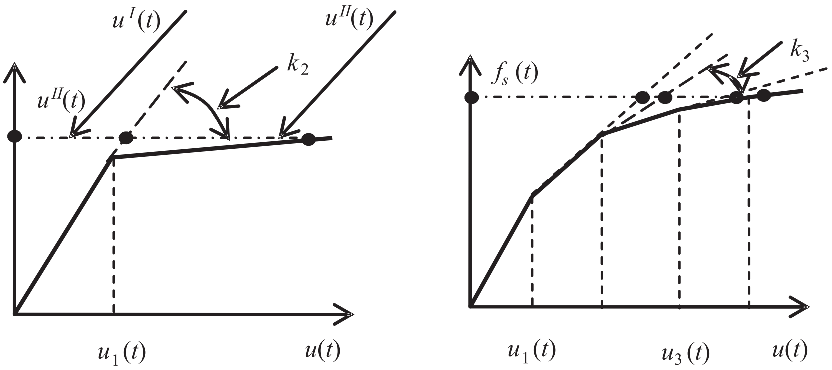

Fig. 3. Illustrations of the displacement decomposition

3. Development of the bi-linear or multi-linear method. The previously presented and proposed in [8] methods have some frailties. The idea of the bi-linear or multi-linear method development is demonstrated in [7]. The solution of eq. (4) is given there as

$$u\,(x) = \frac{k_2}{k_1 \pm k_2} u_1 \,\left(1 - \cos \hat{\omega}_2 t\right) + A \left(\sin \hat{\omega}_1 t + \varphi\right), \tag{8}$$

where $\hat{\omega}_1 = \sqrt{k_1/m}$ and $\hat{\omega}_2 = \sqrt{k_2/m}$. The two terms on the right side of the eq. (8) are illustrated in Fig. 3. Let the terms be denoted as

$$u^I\,(x) = A \left(\sin \hat{\omega}_1 t + \varphi\right) \tag{9}$$

and

$$u^{II}\,(x) = \frac{k_2}{k_1 \pm k_2} u_1 \left(1 - \cos \hat{\omega}_2 t\right). \tag{10}$$

It is not difficult to be shown that if $k_2 = 0$ or $u_1 = 0$, then the dynamic response is transform to linear free vibration and $u\,(x) = u^I\,(x) = A\left(\sin \hat{\omega}_1 t + \varphi\right)$. Now time t_1 related to u_1 can be calculated by

$$t_1 = \frac{1}{\hat{\omega}_2} \arccos \frac{k_1}{\left(k_1 \pm k_2\right) A - k_2 u_1} u_1. \tag{11}$$

The validation of eq. (8) is verified by a numerical example.

4. The harmonic balance method. The method of harmonic balance can be used for nonlinear SDOF free vibration analysis in the case of a general form of the relation$(f_s\,(x) - u\,(x))$. The displacement of the mass point is assumed in a Fourier like series form [9], eq. (2). If eq. (1) is rewritten as

$$\ddot{u}\,(x) + \tilde{f}_s\,(x) = 0, \tag{12}$$

where $\tilde{f}_s\,(x) = f_s\,(x)\,/m$ and using eq. (2) can be obtained a periodic function expressed as

$$\begin{aligned} P\,(x) &= \tilde{f}_s\,(A_0 + A_1 \cos\tilde{\omega}\,t + B_1 \sin\tilde{\omega}\,t + A_2 \cos 2\tilde{\omega}\,t + B_2 \sin 2\tilde{\omega}\,t + \cdots) \\ &- \left[\tilde{\omega}^2\,(A_1 \cos\tilde{\omega}\,t + B_1 \sin\tilde{\omega}\,t) + 4\tilde{\omega}^2\,(A_2 \cos 2\tilde{\omega}\,t + B_2 \sin 2\tilde{\omega}\,t) + \cdots\right] \end{aligned} \tag{13}$$

or

$$\begin{aligned} P\,(x) &= \tilde{f}_s\,(A_0 + A_1 \cos\tilde{\omega}\,t + B_1 \sin\tilde{\omega}\,t + A_2 \cos 2\tilde{\omega}\,t + B_2 \sin 2\tilde{\omega}\,t + \cdots) \\ &- \left[\sum_{i=1}^{\infty} (i.\tilde{\omega})^2\,(A_i \cos\tilde{\omega}\,t + B_i \sin\tilde{\omega}\,t)\right]. \end{aligned} \tag{14}$$

Equation (14), based on an infinite number of terms, can be limited to an appropriate number n in accordance with referred accuracy, written here as

$$\begin{aligned} P\,(x) &= \tilde{f}_s\left(A_0 + \sum_{i=1}^{n} (A_i \cos i\tilde{\omega}\,t + B_i \sin i\tilde{\omega}\,t)\right) \\ &- \left[\sum_{i=1}^{n} (i.\tilde{\omega})^2\,(A_i \cos\tilde{\omega}\,t + B_i \sin\tilde{\omega}\,t)\right]. \end{aligned} \tag{15}$$

The dynamic equilibrium is in existence when the periodic function $P\,(x) = 0$. This can be achieved by using the following equations:

$$\int_0^{2\pi/\tilde{\omega}} \tilde{f}_s(x)\,dt = 0; \quad \tilde{\omega}^2 A_1 = \frac{\tilde{\omega}}{\pi}\int_0^{2\pi/\tilde{\omega}} \tilde{f}_s(x)\cos\tilde{\omega}t\,dt;$$
$$(i\tilde{\omega})^2\,A_i = \frac{\tilde{\omega}}{\pi}\int_0^{2\pi/\tilde{\omega}} \tilde{f}_s(x)\cos i.\tilde{\omega}t\,dt; \quad (i\tilde{\omega})^2 B_i = \frac{\tilde{\omega}}{\pi}\int_0^{2\pi/\tilde{\omega}} \tilde{f}_s(x)\sin i.\tilde{\omega}t\,dt.$$

It is easy to be demonstrated [10] that in the case of linear SDOF free vibration eq. (13) or eq. (14) is transformed to well-known dynamic equilibrium equation

$$\begin{aligned} &P\ (x) = \tilde{f}_s\left(A_0 + \sum_{i=1}^{n} A_i \cos\ i.\tilde{\omega}\ t + B_i \sin\ i.\tilde{\omega}\ t\right) \\ &- \left[\sum_{i=1}^{n} i.\tilde{\omega}^2\,(A_i \cos\ \tilde{\omega}\ t + B_i \sin\ \tilde{\omega}\ t)\right] = \tilde{f}_s\,(t) - \tilde{\omega}^2 u\ (t) \equiv 0. \end{aligned} \tag{16}$$

T a b l e 1

Results for bi-linear elastic force-displacement relation

$T = 0.703$ s	by eq. (4)	for $u^{\max} = 0.2$ m
$T = 0.725$ s	by eq. (4)	for $u^{\max} = 0.3$ m
$T = 0.708$ s	by ANSYS	for $u^{\max} = 0.2$ m
$T = 0.729$ s	by ANSYS	for $u^{\max} = 0.3$ m

Such an operation corresponds with the Fourier series presentation of periodic linear function.

The application of the method of harmonic balance to the analysis of harmonic force vibration with cubic elastic parameter is given in [8]. The equation of motion is written as

$$m\ddot{u}\,(x) + c\dot{u}\,(x) + k\,u\,(x) + \gamma\,u^3\,(x) = \bar{F}\cos\,(\theta\,t + \varphi)\,. \tag{17}$$

and corresponds to cubic increasing of stiffness. If cubic decreasing the stiffness is simulated, eq. (14) must be written in the form

$$m\ddot{u}\,(x) + c\dot{u}\,(x) + k\,u\,(x) + \gamma\sqrt[3]{u}\,(x) = \bar{F}\cos\,(\theta\,t + \varphi)\,. \tag{18}$$

The procedure described by eqs. from (13) to (15) for finite series, limited to number n, can be implemented to MATH calculation product using Fourier series to be analyzed any nonlinear SDOF free vibration. Fourier series applications in the structural mechanics are given in [6, 10] and [7, 11, 12].

5. Numerical example. The numerical example, the results of which are obtained by using bi-linear and three-linear elastic force-displacement relations are compared with the results obtained by FEM computational model. Equation (4) and eq.(5), and ANSYS structural module are used, respectively. Detailed SDOF model information is listed below. The comparison is given in a table form for $u^{\max} = 0.2$ m and $u^{\max} = 0.3$ m. The linear elastic free vibration period is $T_{el} = 0.623$ s.

T a b l e 2

Results for three-linear elastic force-displacement relation

$T = 0.811$ s	by eq. (5)	for $u^{\max} = 0.2$ m
$T = 0.993$ s	by eq. (5)	for $u^{\max} = 0.3$ m
$T = 0.819$ s	by ANSYS	for $u^{\max} = 0.2$ m
$T = 1.004$ s	by ANSYS	for $u^{\max} = 0.3$ m

$$k_1 = 1 \times 10^4 \text{ kN/m}; \quad k_2 = 0.4 \times 10^4 \text{ kN/m}; \quad k_3 = 0.5 \times 10^4 \text{ kN/m};$$

$$m = 100 \text{ kN.s}^2\text{/m}; \quad u_1 = 1 \times 10^{-1} \text{ m}; \quad u_2 = 1.8 \times 10^{-1} \text{ m}.$$

Obviously, the technique, based on eq. (4) and eq.(5) is applicable in the cases, when the displacement $u^{\max}$ is known in advance.

6. Conclusions. This research is devoted to a summarization and as a result of that a discussion on the approaches to nonlinear SDOF free vibration behaviour to be analyzed. In a general form, the equation of motion, can be based on a Fourier like series form. This form can be used in cases of general form of the elastic force-displacement relation. Bi-linear and multi-linear methods are compared with the so-called method of the harmonic balance just from theoretical point of view. The numerical example shows that the results obtained through the discussed methods, bi-linear and multi-linear methods are in a good agreement with the results obtained by a FEM model.

This paper demonstrates a non-standard approach to the eigenvalues and eigenvectors of nonlinear elastic systems to be evaluated. The basis, in which the eigenvalues and eigenvectors are included, can be used for detailed harmonic, periodic, transient or spectral analysis. For application of the method, the ultimate displacements of the system mass points, must be known in advance. This allows the real periods of vibration to be taken into account. The alternative methods use linear elastic relations, multiplied by coefficients or time history analysis.

Generally, the proposed method is applicable in the case of preliminary known or predicted total mass displacements. Such an approach gives a powerful method, the system response to be evaluated in accordance with the real site conditions and Soil-structure interaction.

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