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CATEGORY EQUIVALENCES OF CLONES OF OPERATIONS PRESERVING UNARY OPERATIONS

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Abstract

Any clones on arbitrary set A can be written of the form $\mathrm{Pol}_A Q$ for some set Q of relations on A. We consider clones of the form $\mathrm{Pol}_A Q$ where Q is a set of unary relations on a finite set A. A clone $\mathrm{Pol}_A Q$ is said to be a clone on a set of the smallest cardinality with respect to category equivalence if $|A| \leq |S|$ for all finite sets S and all clones C on S that category equivalent to $\mathrm{Pol}_A Q$. We characterize the clones on a set of the smallest cardinality with respect to category equivalent and show how we can find a clone on a set of the smallest cardinality that category equivalent to a given clone.

Key words: category equivalence of clones, clone of operations
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1. Notations and preliminaries. Let A be a set and let n be a positive integer. The set of all n-ary operations on A is denoted by $O_A^{(n)}$. and the set of all n-ary relations on A is denoted by $\mathrm{Rel}_A^{(n)}$. Let $O_A := \bigcup_{n\geq 1} O_A^{(n)}$ and let $\mathrm{Rel}_A := \bigcup_{n\geq 1} \mathrm{Rel}_A^{(n)}$. A clone on A is a set of operations defined on A which

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contains all projections and is closed under composition. The set of all clones on A is denoted by Clone (A) and it is well-known that Clone (A) forms a complete lattice under inclusion. In the case that $|A| = 2$, the lattice was described by E. Post. For the case $|A| > 2$, it is still an open problem. We want to use the concept of category equivalence of clones to describe elements in Clone (A). In this work we consider clones of the form $\mathrm{Pol}_A Q$ where $Q \subseteq \mathrm{Rel}_A^{(1)}$.

For each $f \in O_A^{(n)}$ and $\rho \in \mathrm{Rel}_A^{(h)}$, we say that f *preserves* ρ if $(f(a_1^1, \ldots, a_1^n), \ldots, f(a_h^1, \ldots, a_h^n)) \in \rho$ whenever $(a_1^1, \ldots, a_h^1), \ldots, (a_1^n, \ldots, a_h^n) \in \rho$. For each $Q \subseteq \mathrm{Rel}_A$ and $F \subseteq O_A$, the set of all n-ary operations on A which preserve all relations in Q is denoted by $\mathrm{Pol}_A^{(n)} Q$, and, the set of all n-ary relations on A, which are preserved by all operations in F, is denoted by $\mathrm{Inv}_A^{(n)} F$. Let $\mathrm{Pol}_A Q := \bigcup_{n \geq 1} \mathrm{Pol}_A^{(n)} Q$ and $\mathrm{Inv}_A F := \bigcup_{n \geq 1} \mathrm{Inv}_A^{(n)} F$.

The set Rel_A together with the operations $\zeta, \tau, \Delta, \circ,$ and $\delta_A^{(1,2;3)}$ form an algebra $\mathbf{Rel_A}$. This algebra is called the *full relation algebra* on A. Each subalgebra of the full relation algebra is called a *relation algebra* and its universe is called a co-clone. It is well-known that $\mathrm{Pol}_A Q$ is a clone on A and $\mathrm{Inv}_A F$ is a co-clone on A for all $Q \subseteq Rel_A$ and $F \subseteq O_A$ and the pair $(\mathrm{Pol}_A, \mathrm{Inv}_A)$ forms a Galois-connection. For more information see [5, 6].

Each class of algebras of the same type can be regarded as a category whose objects are algebras in the class and whose morphisms are homomorphisms between algebras in the class. A variety is a class of algebras of the same type which is closed under operators H, S and P. Therefore, concepts from category theory can be applied to varieties. Two varieties $\mathcal{V}$ and $\mathcal{W}$ are category equivalent, written by $\mathcal{V} \stackrel{\sim}{\equiv} \mathcal{W}$, if there is an equivalence functor from $\mathcal{V}$ to $\mathcal{W}$. For more information see [1, 2] and [7].

The variety generated by an algebra $\mathbf{A}$ is denoted by $V(\mathbf{A})$. The clone generated by the set of all fundamental operations of an algebra $\mathbf{A}$ is denoted by $T(\mathbf{A})$. The set of all subuniverses of $\mathbf{A}$ together with an empty set is denoted by $\mathrm{Sub}\,\mathbf{A}$ and it forms a complete lattice under inclusion. For each $a \in A$, the smallest subuniverse of an algebra $\mathbf{A}$ containing a is denoted by $\langle a \rangle_{\mathrm{Sub}\mathbf{A}}$. In [2], B. A. Davey and H. Werner show that if F is an equivalence functor from a variety $\mathcal{V}$ to a variety $\mathcal{W}$, then the subalgebra lattices of $\mathbf{A}$ and of $F(\mathbf{A})$ are isomorphic for all $\mathbf{A} \in \mathcal{V}$.

In [4], K. Denecke and O. Lüders consider the concept of category equivalences of clones by using the concept of category equivalences of varieties which are generated by one algebra and characterized them by using isomorphisms between relation algebras in [3].

Definition 1.1 ([4])**.** A clone C on a set A is category equivalent to a clone C' on a set B; written by $C \stackrel{\sim}{\equiv} C'$, if there are algebras $\mathbf{A}$ and $\mathbf{B}$ with universes A and B, respectively, such that $C = T(\mathbf{A})$, $C' = T(\mathbf{B})$ and varieties

$V(\mathbf{A})$ and $V(\mathbf{B})$ are category equivalent by an equivalence functor which maps $\mathbf{A}$ to $\mathbf{B}$.

For $Q \subseteq \mathrm{Rel}_A$, we say that a unary operation σ on A is *idempotent* in $\mathrm{Pol}_A Q$ if $\sigma \in \mathrm{Pol}_A Q$ and $\sigma(\sigma(a)) = \sigma(a)$ for all $a \in A$ and is *invertible* in $\mathrm{Pol}_A Q$ if $\sigma \in \mathrm{Pol}_A Q$ and there are a positive integer n, $t \in \mathrm{Pol}_A^{(n)} Q$, and $g_1, \dots, g_n \in \mathrm{Pol}_A^{(1)} Q$ such that $t(\sigma(g_1(a)), \dots, \sigma(g_n(a))) = a$ for all $a \in A$.

For $\rho \in Rel_A^{(h)}$, $Q \subseteq Rel_A$, and a unary operation σ on A, one can define an h-ary relation $\sigma(\rho) := \rho \cap (\sigma(A))^h$ and a set $\sigma(Q) := \{\sigma(\rho) \mid \rho \in Q\}$.

Proposition 1.2 ([4]). *Let A be a finite set, let $Q \subseteq Rel_A$, and let σ be invertible idempotent in $\mathrm{Pol}_A Q$. Then $\mathrm{Pol}_A Q$ and $\mathrm{Pol}_{\sigma(A)} \sigma(Q)$ are category equivalent.*

Theorem 1.3 ([3]). *Two clones C and C' on finite sets A and B, respectively, are category equivalent if and only if the relation algebras $\mathbf{Inv_A C}$ and $\mathbf{Inv_B C'}$ are isomorphic.*

2. Main results. For any finite set A with at least two elements and any subset Q of $Rel_A^{(1)}$, we have that $\mathrm{Pol}_A(Q \setminus \{A\}) = \mathrm{Pol}_A(Q \setminus \{\varnothing\}) = \mathrm{Pol}_A Q$ since $\mathrm{Pol}_A\{A\} = \mathrm{Pol}_A\{\varnothing\} = O_A$. So, we will ignore the cases that Q contains A or $\varnothing$. For each positive integer n, let $\tilde{n} := \{0, 1, \dots, n-1\}$. First, we mention important observations.

Proposition 2.1. *Let A and B be finite sets with at least two elements, let n be a positive integer, let $A_1, \dots, A_n$ be nonempty proper subsets of A, and let C be a clone on B. If $C \overset{\sim}{\equiv} \mathrm{Pol}_A\{A_1, \dots, A_n\}$, then there are nonempty proper subsets $B_1, \dots, B_n$ of B such that $C = \mathrm{Pol}_B\{B_1, \dots, B_n\}$ and $|B_i| = 1$ if and only if $|A_i| = 1$ for all $1 \le i \le n$.*

Proof. If $C \overset{\sim}{\equiv} \mathrm{Pol}_A\{A_1, \dots, A_n\}$, then there is an isomorphism φ from the relation algebra $\mathbf{Inv_A Pol_A}\{\mathbf{A_1}, \dots, \mathbf{A_n}\}$ onto the relation algebra $\mathbf{Inv_B C}$. Thus $C = \mathrm{Pol}_B \,\mathrm{Inv}_B C = \mathrm{Pol}_B \varphi(\mathrm{Inv}_A \mathrm{Pol}_A\{A_1, \dots, A_n\}) = \mathrm{Pol}_B \,\mathrm{Inv}_B \mathrm{Pol}_B\ \{\varphi(A_1), \dots, \varphi(A_n)\} = \mathrm{Pol}_B\{\varphi(A_1), \dots, \varphi(A_n)\}$. Moreover, $\varphi(A_1), \dots, \varphi(A_n)$ are nonempty proper subsets of B and $|\varphi(A_i)| = 1$ if and only if $|A_i| = 1$ for all $1 \le i \le n$. □

Proposition 2.2. *Let Q be a set of nonempty proper subsets of a finite set A where $|A| \ge 2$ and let $\mathbf{A}$ be a finite algebra with the universe A. If $T(\mathbf{A}) = \mathrm{Pol}_A Q$, then $\mathrm{Sub}\,\mathbf{A} = \{A\} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q\}$.*

Proof. If $S \in \mathrm{Sub}\,\mathbf{A}$ where $S \ne A$ and $S \ne \varnothing$, then $S = \bigcap\{B \mid B \in Q \text{ and } S \subseteq B\}$ Thus $\mathrm{Sub}\,\mathbf{A} \subseteq \{A\} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q\}$.

Conversely, if $B \in Q$, then $B \in \mathrm{Sub}\,\mathbf{A}$. This implies that $\bigcap \mathcal{B} \in \mathrm{Sub}\,\mathbf{A}$ for all $\mathcal{B} \subseteq Q$. Then $\{A\} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q\} \subseteq \mathrm{Sub}\,\mathbf{A}$. □

The following propositions present clones on a set of the smallest cardinality that category equivalent to clones $\mathrm{Pol}_A Q$ where $Q \subseteq Rel_A^{(1)}$ and $1 \le |Q| \le 2$.

Proposition 2.3 ([4]). *Let C be a clone on a finite set A where $|A| \geq 2$. The following statements are fulfilled:*

1. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{2}}\,\{\{0\}\}$ *if and only if* $C = \mathrm{Pol}_A\,\{\{a\}\}$ *for some* $a \in A$.

2. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{3}}\,\{\{0,1\}\}$ *if and only if* $C = \mathrm{Pol}_A\{B\}$ *for some proper subset* B *of* A *such that* $|B| \geq 2$.

3. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{2}}\,\{\{0\},\{1\}\}$ *if and only if* $C = \mathrm{Pol}_A\,\{\{a\},\{b\}\}$ *for some* $a, b \in A$.

Proposition 2.4. *Let C be a clone on a finite set A where $|A| \geq 2$. The following statements are fulfilled:*

1. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{4}}\,\{\{0,1\},\{2,3\}\}$ *if and only if* $C = \mathrm{Pol}_A\{B_1, B_2\}$ *for some* $B_1, B_2 \subsetneqq A$ *such that* $|B_1| \geq 2$, $|B_2| \geq 2$ *and* $B_1 \cap B_2 = \varnothing$.

2. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{4}}\,\{\{0,1\},\{0,1,2\}\}$ *if and only if* $C = \mathrm{Pol}_A\{B_1, B_2\}$ *for some* $B_1, B_2 \subsetneqq A$ *such that* $|B_1| \geq 2$ *and* $B_1 \subsetneqq B_2$.

3. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{3}}\,\{\{0,1\},\{1,2\}\}$ *if and only if* $C = \mathrm{Pol}_A\{B_1, B_2\}$ *for some* $B_1, B_2 \subsetneqq A$ *such that* $B_1 \not\subseteq B_2$, $B_2 \not\subseteq B_1$ and $|B_1 \cap B_2| = 1$.

4. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{4}}\,\{\{0,1,2\},\{1,2,3\}\}$ *if and only if* $C = \mathrm{Pol}_A\{B_1, B_2\}$ *for some* $B_1, B_2 \subsetneqq A$ *such that* $B_1 \not\subseteq B_2$, $B_2 \not\subseteq B_1$ *and* $|B_1 \cap B_2| \geq 2$.

5. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{3}}\,\{\{0\},\{1,2\}\}$ *if and only if* $C = \mathrm{Pol}_A\,\{\{a\}, B\}$ *for some* $a \in A$ *and* $B \subseteq A \setminus \{a\}$ *such that* $|B| \geq 2$.

6. $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{3}}\,\{\{1\},\{1,2\}\}$ *if and only if* $C = \mathrm{Pol}_A\,\{\{a\}, B\}$ *for some* $a \in B \subsetneqq A$ *such that* $|B| \geq 2$.

Proof. We show (1), the proofs for (2)–(7) are similar.

(1) If $C \overset{\sim}{\equiv} \mathrm{Pol}_{\tilde{4}}\,\{\{0,1\},\{2,3\}\}$, then there are algebras $\mathbf{4}$ and $\mathbf{A}$ with universes $\tilde{4}$ and A, respectively, such that $T(\mathbf{4}) = \mathrm{Pol}_{\tilde{4}}\,\{\{0,1\},\{2,3\}\}$ and $T(\mathbf{A}) = C$ and the varieties $V(\mathbf{4})$ and $V(\mathbf{A})$ are category equivalent by an equivalence functor which maps $\mathbf{4}$ to $\mathbf{A}$. Thus $(\mathrm{Sub}\,\mathbf{4}; \subseteq) \cong (\mathrm{Sub}\,\mathbf{A}; \subseteq)$, $\mathrm{Sub}\,\mathbf{4} = \{\varnothing, \tilde{4}, \{0,1\}, \{2,3\}\}$, and $T(\mathbf{A}) = \mathrm{Pol}_A\{B_1, B_2\}$ for some $B_1, B_2 \subsetneqq A$ such that $|B_1| \geq 2$, $|B_2| \geq 2$. Hence, $\mathrm{Sub}\,\mathbf{A} = \{\varnothing, A, B_1, B_2\}$ and $B_1 \cap B_2 = \varnothing$.

Conversely, assume that $A = \{0, 1, \dots, n-1\}$ where $n \geq 4$ and $0, 1 \in B_1$ and $2, 3 \in B_2$. Let σ be a unary operation on A which is defined by

$$\sigma(x) := \begin{cases} x, & 0 \leq x \leq 3; \\ 2, & x \in B_2 \setminus \{2,3\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then σ is a unary invertible idempotent operation in $\mathrm{Pol}_A\{B_1, B_2\}$. By Proposition 1.2, we have that $C = \mathrm{Pol}_A\{B_1, B_2\} \overset{\sim}{\equiv} \mathrm{Pol}_{\sigma(A)}\{\sigma(B_1), \sigma(B_2)\} = \mathrm{Pol}_{\tilde{4}}\{\{0,1\},\{2,3\}\}$. □

For any set Q of nonempty proper subsets of A, we can find a clone on a set of the smallest cardinality that category equivalent to the clone $\mathrm{Pol}_A Q$ by stepwise reduction of the cardinality of A using the following theorems.

Theorem 2.3. *Let Q be a set of nonempty proper subsets of a finite set A where $|A| \geq 2$ and let $a \in A$. If $a \notin \bigcup Q$, then $\mathrm{Pol}_A Q \overset{\sim}{\equiv} \mathrm{Pol}_{A\setminus\{a\}} Q$ if and only if $A \setminus \{a\} \notin Q$.*

Proof. If $\mathrm{Pol}_A Q \overset{\sim}{\equiv} \mathrm{Pol}_{A\setminus\{a\}} Q$, then there are algebras $\mathbf{A}$ and $\mathbf{A} \setminus \{\mathbf{a}\}$ such that $T(\mathbf{A}) = \mathrm{Pol}_A Q$ and $T(\mathbf{A} \setminus \{\mathbf{a}\}) = \mathrm{Pol}_{A\setminus\{a\}} Q$ and the varieties $V(\mathbf{A})$ and $V(\mathbf{A} \setminus \{\mathbf{a}\})$ are category equivalent by an equivalence functor which maps $\mathbf{A}$ to $\mathbf{A} \setminus \{\mathbf{a}\}$. Thus $(\mathrm{Sub}\,\mathbf{A}; \subseteq) \cong (\mathrm{Sub}\,(\mathbf{A} \setminus \{\mathbf{a}\}); \subseteq)$ and consequencely $|\mathrm{Sub}\,\mathbf{A}| = |\mathrm{Sub}\,(\mathbf{A} \setminus \{\mathbf{a}\})|$.

Assume that $A \setminus \{a\} \in Q$. Let $\mathcal{A} := \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q\}$ and $\mathcal{A}_a := \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q \setminus \{A \setminus \{a\}\}\}$. Since $T(\mathbf{A}) = \mathrm{Pol}_A Q$, we have $\mathrm{Sub}\,\mathbf{A} = \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q\} \cup \{A\} = \mathcal{A} \cup \{A\}$. Since $T(\mathbf{A} \setminus \{\mathbf{a}\}) = \mathrm{Pol}_{A\setminus\{a\}} Q = \mathrm{Pol}_{A\setminus\{a\}}(Q \setminus \{A \setminus \{a\}\})$, we have $\mathrm{Sub}\,(\mathbf{A} \setminus \{\mathbf{a}\}) = \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q \setminus \{A \setminus \{a\}\}\} \cup \{A \setminus \{a\}\} = \mathcal{A}_a \cup \{A \setminus \{a\}\} = \mathcal{A}$. Then $|\mathrm{Sub}\,\mathbf{A}| > |\mathrm{Sub}\,(\mathbf{A} \setminus \{\mathbf{a}\})|$. This is a contradiction. Therefore $A \setminus \{a\} \notin Q$.

Conversely, assume that $A \setminus \{a\} \notin Q$. Then $|A \setminus \{a\}| \geq 2$. For $b \in A \setminus \{a\}$, let $\sigma : A \to A$ be defined by $\sigma(x) := x$ if $x \neq a$ and $\sigma(x) := b$ if $x = a$. Then σ is a unary invertible idempotent operation in $\mathrm{Pol}_A Q$ which implies that $\mathrm{Pol}_A Q \overset{\sim}{\equiv} \mathrm{Pol}_{\sigma(A)} \sigma(Q) = \mathrm{Pol}_{A\setminus\{a\}} Q$. □

For each $a \in A$ and each $Q \subseteq Rel_A^{(1)}$, let $Q_a := \{B \setminus \{a\} \mid B \in Q\}$ and let $\mathcal{B}_a := \{B \mid B \in Q \text{ and } a \in B\}$.

Theorem 2.4. *Let Q be a set of nonempty proper subsets of a finite set A where $|A| \geq 2$ and let $a \in A$. If $a \in \bigcup Q$, then $\mathrm{Pol}_A Q \overset{\sim}{\equiv} \mathrm{Pol}_{A\setminus\{a\}} Q_a$ if and only if $|\bigcap \mathcal{B}_a| \geq 3$ and $(\bigcap \mathcal{B}_a) \setminus \{a\} \nsubseteq B$ for all $B \in Q \setminus \mathcal{B}_a$.*

Proof. If $\mathrm{Pol}_A Q \overset{\sim}{\equiv} \mathrm{Pol}_{A\setminus\{a\}} Q_a$, then there are algebras $\mathbf{A}$ and $\mathbf{A} \setminus \{\mathbf{a}\}$ such that $T(\mathbf{A}) = \mathrm{Pol}_A Q$ and $T(\mathbf{A} \setminus \{\mathbf{a}\}) = \mathrm{Pol}_{A\setminus\{a\}} Q_a$ and varieties $V(\mathbf{A})$ and

$V(\mathbf{A} \setminus \{\mathbf{a}\})$ are category equivalent by an equivalence functor which maps $\mathbf{A}$ to $\mathbf{A} \setminus \{\mathbf{a}\}$. Thus $(\operatorname{Sub}\mathbf{A}; \subseteq) \cong (\operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\}); \subseteq)$ and consequently $|\operatorname{Sub}\mathbf{A}| = |\operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\})|$.

Let $\gamma : \operatorname{Sub}\mathbf{A} \to \operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\})$ be defined by $\gamma(S) := S \setminus \{a\}$. Then γ is surjective. Thus $\gamma(\operatorname{Sub}\mathbf{A}) = \operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\})$.

Assume that there is $B \in Q \setminus \mathcal{B}_a$ such that $(\bigcap \mathcal{B}_a) \setminus \{a\} \subseteq B$. We have that $\bigcap \mathcal{B}_a \in \operatorname{Sub}\mathbf{A}$, $B \cap (\bigcap \mathcal{B}_a) \in \operatorname{Sub}\mathbf{A}$, and $\gamma(B \cap (\bigcap \mathcal{B}_a)) = (B \cap (\bigcap \mathcal{B}_a)) \setminus \{a\} = (B \setminus \{a\}) \cap ((\bigcap \mathcal{B}_a) \setminus \{a\}) = B \cap ((\bigcap \mathcal{B}_a) \setminus \{a\}) = (\bigcap \mathcal{B}_a) \setminus \{a\} = \gamma(\bigcap \mathcal{B}_a)$ but $B \cap (\bigcap \mathcal{B}_a) \neq \bigcap \mathcal{B}_a$. Then γ is not injective. Thus $|\operatorname{Sub}\mathbf{A}| > |\gamma(\operatorname{Sub}\mathbf{A})|$. Hence $|\operatorname{Sub}\mathbf{A}| > |\operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\})|$. This is a contradiction. Therefore $(\bigcap \mathcal{B}_a) \setminus \{a\} \not\subseteq B$ for all $B \in Q \setminus \mathcal{B}_a$.

Assume that $|\bigcap \mathcal{B}_a| = 1$. We have that $\{a\} = \bigcap \mathcal{B}_a \in \operatorname{Sub}\mathbf{A}$, $\varnothing \in \operatorname{Sub}\mathbf{A}$, and $\gamma(\{a\}) = \varnothing = \gamma(\varnothing)$ but $\{a\} \neq \varnothing$. Then γ is not injective. Thus $|\operatorname{Sub}\mathbf{A}| > |\gamma(\operatorname{Sub}\mathbf{A})|$. Hence $|\operatorname{Sub}\mathbf{A}| > |\operatorname{Sub}(\mathbf{A} \setminus \{\mathbf{a}\})|$. This is a contradiction. Therefore $|\bigcap \mathcal{B}_a| \geq 2$.

Assume that $|\bigcap \mathcal{B}_a| = 2$. Since $T(\mathbf{A}) \overset{\sim}{\equiv} T(\mathbf{A} \setminus \{\mathbf{a}\})$, there is an isomorphism φ from $\mathbf{Inv_A T}(\mathbf{A})$ onto $\mathbf{Inv_{A \setminus \{a\}} T}(\mathbf{A} \setminus \{\mathbf{a}\})$. Then $\operatorname{Pol}_{A\setminus\{a\}} Q_a = \operatorname{Pol}_{A\setminus\{a\}} \varphi(Q)$. Thus $\{\bigcap \mathcal{B}' \mid \mathcal{B}' \subseteq Q_a\} = \{\bigcap \mathcal{B}' \mid \mathcal{B}' \subseteq \varphi(Q)\}$. Let $\mathcal{A}_1 := \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq Q$ and $|\bigcap \mathcal{B}| \geq 2\}$, let $\mathcal{A}_1' := \mathcal{A}_1 \setminus \{\bigcap \mathcal{B}_a\}$, and let $\mathcal{A}_2 := \{\bigcap \mathcal{B}' \mid \mathcal{B}' \subseteq Q_a$ and $|\bigcap \mathcal{B}'| \geq 2\}$. Then $|\mathcal{A}_1| > |\mathcal{A}_1'|$. Let $\gamma_1 : \mathcal{A}_1 \to \mathcal{A}_2$ be defined by $\gamma_1(\bigcap \mathcal{B}) := \bigcap \varphi(\mathcal{B})$ and let $\gamma_2 : \mathcal{A}_1' \to \mathcal{A}_2$ be defined by $\gamma_2(\bigcap \mathcal{B}) := (\bigcap \mathcal{B}) \setminus \{a\}$. Then γ_1 is bijective and γ_2 is surjective. Thus $|\mathcal{A}_1| = |\mathcal{A}_2|$ and $|\mathcal{A}_1'| \geq |\mathcal{A}_2|$. Hence $|\mathcal{A}_2| = |\mathcal{A}_1| > |\mathcal{A}_1'| \geq |\mathcal{A}_2|$. This is a contradiction. Therefore, $|\bigcap \mathcal{B}_a| \geq 3$.

Conversely, assume that $|\bigcap \mathcal{B}_a| \geq 3$ and $(\bigcap \mathcal{B}_a) \setminus \{a\} \not\subseteq B$ for all $B \in Q \setminus \mathcal{B}_a$. For $b \in (\bigcap \mathcal{B}_a) \setminus \{a\}$, let $\sigma : A \to A$ be defined by $\sigma(x) := x$ if $x \neq a$ and $\sigma(x) := b$ if $x = a$. We have that σ is a unary invertible idempotent operation in $\operatorname{Pol}_A Q$. Then $\operatorname{Pol}_A Q \overset{\sim}{\equiv} \operatorname{Pol}_{\sigma(A)} \sigma(Q) = \operatorname{Pol}_{A\setminus\{a\}} Q_a$. □

Now we can proof our main result.

Theorem 2.5. *Let Q be a set of nonempty proper subsets of a finite set A where $|A| \geq 2$. The clone $\operatorname{Pol}_A Q$ is a clone on a set of the smallest cardinality with respect to category equivalence if and only if*

1. *if $a \in A \setminus \bigcup Q$, then $A \setminus \{a\} \in Q$;*
2. *if $a \in \bigcup Q$, then $|\bigcap \mathcal{B}_a| \leq 2$ or $(\bigcup \mathcal{B}_a) \setminus \{a\} \subseteq B$ for some $B \in Q \setminus \mathcal{B}_a$.*

Proof. If $\operatorname{Pol}_A Q$ is a clone on a set of the smallest cardinality with respect to category equivalence, then we get (1) and (2) by Theorem 2.3 and Theorem 2.4, respectively.

Conversely, let S be a finite set with at least two elements and let Q' be a set of nonempty proper subsets of S such that $\operatorname{Pol}_A Q \overset{\sim}{\equiv} \operatorname{Pol}_S Q'$. Then there are algebras $\mathbf{A}$ and $\mathbf{S}$ such that $T(\mathbf{A}) = \operatorname{Pol}_A Q$, $T(\mathbf{S}) = \operatorname{Pol}_S Q'$ and the varieties $V(\mathbf{A})$ and $V(\mathbf{S})$ are category equivalent by an equivalence functor which maps $\mathbf{A}$ to $\mathbf{S}$.

For $Z \in \{A, S\}$, let $X_Z := \bigcup \mathcal{X}_Z$ where $\mathcal{X}_Z$ is the set of all atoms in $\operatorname{Sub}\mathbf{Z}$, let $Y_Z := \{\langle a\rangle_{\operatorname{Sub}\mathbf{Z}} \mid a \in A \setminus X_Z\}$, and let $\gamma_Z : Z \to X_Z \cup Y_Z$ be defined by $\gamma_Z(x) := x$ if $x \in X_Z$ and $\gamma_Z(x) := \langle x\rangle_{\operatorname{Sub}\mathbf{Z}}$ if $x \notin X_Z$. It is easy to see that γ_Z is surjective. By (1) and (2), one can show that γ_A is injective. Then $|A| = |X_A \cup Y_A| = |X_A| + |Y_A|$ and $|S| \geq |X_S \cup Y_S| = |X_S| + |Y_S|$.

Since $\operatorname{Pol}_A Q \overset{\sim}{\equiv} \operatorname{Pol}_S Q'$, there is an isomorphism φ from relation algebra $\mathbf{Inv_A\,Pol_A\,Q}$ onto relation algebra $\mathbf{Inv_S\,Pol_S\,Q'}$. Moreover, (i) $|B| = 1$ if and only if $|\varphi(B)| = 1$ for all $B \in \operatorname{Inv}_A\operatorname{Pol}_A Q$, (ii) $\varphi_{|\operatorname{Sub}\mathbf{A}}$ is a lattice-isomorphism from $(\operatorname{Sub}\mathbf{A}; \subseteq)$ onto $(\operatorname{Sub}\mathbf{S}; \subseteq)$, (iii) $\varphi_{|\mathcal{X}_A}$ is a bijection from $\mathcal{X}_A$ onto $\mathcal{X}_S$, and (iv) $\varphi_{|Y_A}$ is an injection from Y_A onto Y_S.

For each $B \in \mathcal{X}_A$, we have that $|B| \leq 2$, and then $|B| \leq |\varphi(B)|$. So $|X_A| = \left|\bigcup \mathcal{X}_A\right| = \sum_{B\in\mathcal{X}_A} |B| \leq \sum_{B\in\mathcal{X}_A} |\varphi(B)| = \sum_{D\in\mathcal{X}_S} |D| = \left|\bigcup \mathcal{X}_S\right| = |X_S|$. Since $\varphi_{|Y_A}$ is an injection from Y_A onto Y_S, we have that $|Y_A| \leq |Y_S|$. Therefore, $|A| = |X_A| + |Y_A| \leq |X_S| + |Y_S| \leq |S|$. □

A clone that we get by stepwise reduction of the cardinality of A using Theorem 2.3 and Theorem 2.4 satisfies conditions in Theorem 2.5 and this shows that Theorem 2.3 and Theorem 2.4 deliver a method to find a clone on a set of the smallest cardinality that is category equivalent to a given clone.

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