Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 66, No 2, 2013

MATHEMATIQUES
Équations différentielles

APPLICATION OF THE DIFFERENTIAL CALCULUS TO NONLINEAR PARABOLIC OPERATORS

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(*Submitted by Academician P. Popivanov on September 24, 2012*)

Abstract

We consider the Cauchy–Dirichlet problem for second order quasilinear non-divergence form parabolic operators with discontinuous data. Fixing a solution $u_0 \in W_p^{2,1}(Q)$, $p > n+2$ in the coefficients and taking the Fréchet derivative of the operator at u_0 we obtain formally a linear non-degenerative problem. We apply the Implicit Function Theorem in order to show that for all small L^∞-perturbations of the data there exists, locally in time, exactly one solution u close to u_0 in $W_p^{2,1}$, which depends smoothly on the data. Moreover, applying the Newton Iteration Procedure, we obtain an approximating sequence for u_0.

Key words: nonlinear parabolic operators, Cauchy–Dirichlet problem, strong solutions, perturbation, Implicit Function Theorem, Newton Iteration Procedure

2000 Mathematics Subject Classification: 35K15, 35K59, 47J07, 58C15

1. Statement of the problem. We consider the following Cauchy–Dirichlet problem for non-divergent parabolic equation

$$(1)\qquad \begin{cases} D_t u - a^{ij}(x,t,u,Du)D_{ij}u = f(x,t,u,Du) \\ u \in \mathfrak{W}_p(Q) = \left\{ u \in W_p^{2,1}(Q),\ u(x,t) = 0 \text{ on } \partial Q \right\},\ p > n+2 \end{cases}$$

in a cylinder $Q = \Omega \times \mathcal{S}$ with a base $\Omega \subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$ and $\mathcal{S} = (0,T)$. The data are supposed to satisfy the Carathéodory conditions. Along with (1) we consider its formal linearization obtained by derivation in the sense of Fréchet at some fixed solution u_0

$$\text{(2)}\qquad \begin{cases} D_t u - a^{ij}(x,t,u_0,Du_0)D_{ij}u \\ \qquad -\sum_{l=1}^{n}\left[D_{\xi_l}a^{ij}(x,t,u_0,Du_0)D_{ij}u_0 + D_{\xi_l}f(x,t,u_0,Du_0)\right]D_{\xi_l}u \\ \qquad -\left[D_u a^{ij}(x,t,u_0,Du_0)D_{ij}u_0 + D_u f(x,t,u_0,Du_0)\right]u = 0 \\ u\in\mathfrak{W}_p(Q),\ \ p>n+2\,. \end{cases}$$

Assuming that (2) has no non-trivial solutions it becomes a Fredholm operator (index zero) which is an isomorphism from $\mathfrak{W}_p(Q)$ onto $L^p(Q)$. We show that for small L^∞-perturbations of the data, there exists exactly one local in time solution of the perturbed problem which is close to u_0 in the sense of $W_p^{2,1}$ and depends continuously on the perturbing functions $(\{\tilde a^{ij}\},\tilde f)$. Further, for given u_1 we determine a Newton Iteration $\{u_{k+1}\}_{k=1}^{\infty}$ where u_{k+1} is a solution of the *linearized non-homogeneous problem*

$$\begin{cases} D_t u_{k+1} - a^{ij}(x,t,u_k,Du_k)D_{ij}u_{k+1} \\ \qquad -\sum_{l=1}^{n}\left[D_{\xi_l}a^{ij}(x,t,u_k,Du_k)D_{ij}u_k + D_{\xi_l}f(x,t,u_k,Du_k)\right]D_{\xi_l}u_{k+1} \\ \qquad -\left[D_u a^{ij}(x,t,u_k,Du_k)D_{ij}u_k - D_u f(x,t,u_k,Du_k)\right]u_{k+1} \\ = D_t u_k - \sum_{l=1}^{n}\left[D_{\xi_l}a^{ij}(x,t,u_k,Du_k)D_{ij}u_k + D_{\xi_l}f(x,t,u_k,Du_k)\right]D_{\xi_l}u_k \\ \qquad -\left[D_u a^{ij}(x,t,u_k,Du_k)D_{ij}u_k - D_u f(x,t,u_k,Du_k)\right]u_k \\ u_{k+1}\in\mathfrak{W}_p(Q),\ \ p>n+2 \end{cases}$$

for each index $k\ge 1$. We prove that if the initial iteration u_1 is close to u_0 in $W_p^{2,1}$, then the iteration sequence converges to u_0, i.e. $\|u_k-u_0\|_{W_p^{2,1}(Q)}\to 0$ as $k\to\infty$.

Let us note that there are no any growth conditions imposed on $a^{ij}(x,t,u,\xi)$ and $f(x,t,u,\xi)$. However, certain uniform boundedness and continuity of these functions with respect to (u,ξ) are required in order to ensure the superposition operators $u\mapsto a^{ij}(\cdot,\cdot,u(\cdot,\cdot),Du(\cdot,\cdot))$ and $u\mapsto f(\cdot,\cdot,u(\cdot,\cdot),Du(\cdot,\cdot))$ to be C^1-maps from $W_x^{1,\infty}(Q)$ into $L^\infty(Q)$ and $L^p(Q)$, respectively.

Results, as the presented here, hold also for elliptic quasilinear equations in divergence and non-divergence form (see [3–5]).

The corresponding parabolic divergence form equations and weakly coupled systems are studied in [2]. Let us note that in [2] the conditions on the domain are more general (it has to be a set with Lipschitz boundary), but the data of the problem depend only on u. Similar results are also obtained for operators satisfying the Campanato condition. In [1] and [6] a version of the IFT and its application to the study of "near operators" are presented.

2. Application of the implicit function theorem. Introducing the superposition operators

$$\begin{cases} \mathcal{A}_{ij}(u) := a^{ij}(x,t,u,Du), \qquad \mathcal{F}(u) := f(x,t,u,Du) \\ \mathcal{P}(u) := D_t u - \mathcal{A}_{ij}(u) D_{ij} u - \mathcal{F}(u), \end{cases} \tag{3}$$

we can rewrite problem (1) in the form

$$\mathcal{P}(u) = 0, \quad u \in \mathfrak{W}_p(Q),\ p > n+2\,. \tag{4}$$

Fixing a function $u_0 \in \mathfrak{W}_p(Q)$ and taking the Fréchet derivative of $\mathcal{P}(u)$ at u_0 we obtain a *formally linearized problem*

$$\begin{cases} D_u\mathcal{P}(u_0)v = D_t v - \mathcal{A}_{ij}(u_0)D_{ij}v \\ \qquad -\big(D_u\mathcal{A}_{ij}(u_0)D_{ij}u_0 + D_u\mathcal{F}(u_0)\big)v = 0 \\ v \in \mathfrak{W}_p(Q),\ p > n+2, \end{cases} \tag{5}$$

where

$$\begin{cases} D_u\mathcal{A}_{ij}(u) = D_u a^{ij}(x,t,u,Du) + \displaystyle\sum_{l=1}^{n} D_{\xi_l} a^{ij}(x,t,u,Du) D_{\xi_l} \\ D_u\mathcal{F}(u) = D_u f(x,t,u,Du) + \displaystyle\sum_{l=1}^{n} D_{\xi_l} f(x,t,u,Du) D_{\xi_l}. \end{cases} \tag{6}$$

In order to describe the regularity of the data, we need the following definition.

Definition 1. *Let $\mathcal{D} \subseteq \mathbb{R}\times\mathbb{R}^n$ and $a(x,t,u,\xi) : Q\times\mathcal{D} \to \mathbb{R}$ be a Carathéodory function, then it is said to be a $\mathfrak{C}^1$-Carathéodory function if $a(x,t,\cdot,\cdot)$ is continuously differentiable with respect to (u,ξ) and for each compact $K \subset \mathcal{D}$, a, $D_u a$ and $D_{\xi_l} a$ are bounded and uniformly continuous in $(u,\xi) \in K$ for a.a. $(x,t) \in Q$. The vector space of $\mathfrak{C}^1(Q\times K)$-Carathéodory functions is equipped with the norm*

$$\|a\|_{\mathfrak{C}^1(Q\times K)} := \sup_{(\xi,\eta)\in K}\ \operatorname*{esssup}_{(x,t)\in Q} \Big(|a| + |D_u a| + \sum_{l=1}^{n} |D_{\xi_l} a|\Big).$$

The function a is called $\mathfrak{C}^{1,1}$-Carathéodory function in $Q \times D$ if $a \in \mathfrak{C}^1$ and in addition $a, D_u a$ and $D_{\xi_l} a$ are Lipschitz continuous with respect to (u,ξ), i.e. for *each compact* $K \subset D$ *there exists a constant* $L_a > 0$ *such that*

$$|a(x,t,u,\xi) - a(x,t,u',\xi')| + |D_u a(x,t,u,\xi) - D_u a(x,t,u',\xi')|$$
$$+\sum_{l=1}^{n} |D_{\xi_l} a(x,t,u,\xi) - D_{\xi_l} a(x,t,u',\xi')| \le L_a\big(|u-u'| + |\xi - \xi'|\big).$$

Let K and D be as above. The following results are analogous of Lemmata 1 and 2 in [3] and describe the regularity of the superposition operator a.

Lemma 1. *Let* $a : Q \times D \to \mathbb{R}$ *be a Carathéodory function satisfying*

- $a(\cdot,\cdot,u,\xi) \in VMO(Q)$ *locally uniformly in* (u,ξ) *with* VMO*-modulus*

  $$\gamma_K(r) := \sup_{(u,\xi)\in K} \sup_{\mathcal{C}_\rho, \rho \le r} \fint_{Q_\rho} \Big|a(y,\tau,u,\xi) - \fint_{Q_\rho} a(z,\zeta,u,\xi)dz\, d\zeta\Big| dy\, d\tau,$$

  where $Q_\rho = Q \cap \mathcal{C}_\rho$ *and* $\mathcal{C}_\rho$ *ranges over all parabolic cylinders centred in some* $(x,t) \in Q$;

- $a(x,t;\cdot,\cdot)$ *is local uniform continuous, that is, for each compact* $K \subset \mathcal{D}$ *there exists* $C_K > 0$ *and a nondecreasing function* $\mu_K : (0,\infty) \to (0,\infty)$, $\lim_{\omega\to 0} \mu_K(\omega) = 0$ *such that for all* $(u,\xi),(u',\xi') \in K$ *it holds*

  $$|a(x,t,u,\xi) - a(x,t,u',\xi')| \le \mu_K(|u-u'|) + C_K|\xi - \xi'| \quad a.a.\ (x,t) \in Q;$$

- $a_0 = a(x,t,0,0) \in L^\infty(Q)$.

Then for each $u \in W_p^{2,1}(Q)$ *with* $p > n+2$, *the operator* $a(x,t,u(x,t),Du(x,t))$ *is in* $VMO \cap L^\infty(Q)$ *with* VMO*-modulus*

$$\gamma_a(r) = \sup_{\rho \le r} \fint_{Q_\rho} \Big|a(y,\tau,u(y,\tau),Du(y,\tau)) - \fint_{Q_\rho} a(z,\zeta,u(z,\zeta),Du(z,\zeta))dzd\zeta\Big| dyd\tau.$$

Lemma 2. *Let* $a \in \mathfrak{C}^1(Q \times \bar{D})$ *and* $A(a;u) := a(x,t,u,Du)$ *be an evaluation map. Denote*

$$\mathcal{U} = \big\{u \in W_x^{1,\infty}(Q) : (u,Du) \in K$$

then $\mathcal{U}$ *is an open set in* $W_x^{1,\infty}(Q)$ *and*

$$A(a;u) \in C^1(\mathfrak{C}^1(Q \times \bar{\mathcal{D}}) \times \mathcal{U}; L^\infty(Q)).$$

We study problem (1), a subject to the following hypothesis:

$\mathbf{H_1}$) $a^{ij}, f : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $\mathfrak{C}^{1,1}$-Carathéodory functions;

$\mathbf{H_2}$) Let $U \subset C(\bar{\mathcal{S}}, C^1(\bar{\Omega}))$ be an open set and there exists at least one solution $u_0 \in U \cap \mathfrak{W}_p(Q)$, $p > n+2$ of (1);

$\mathbf{H_3}$) There exists a positive constant λ such that

$$\begin{cases} \lambda|\eta|^2 \le a^{ij}(x,t,u_0,Du_0)\eta_i\eta_j \le \lambda^{-1}|\eta|^2 & \text{a.a. } (x,t) \in Q, \ \forall \eta \in \mathbb{R}^n, \\ a^{ij} = a^{ji} \quad \forall\, 1 \le i,j \le n. \end{cases}$$

$\{a^{ij}(x,t,u_0,Du_0)\}_{i,j=1}^n \in VMO \cap L^\infty(Q)$ with VMO-modulus

$\gamma_a(r) = \displaystyle\sum_{ij=1}^n \gamma_{a^{ij}}(r)$ and $f(x,t,u_0,Du_0) \in L^p(Q)$, $p > n+2$;

$\mathbf{H_4}$) There are no non-trivial solutions $v \in \mathfrak{W}_p(Q)$, $p > n+2$ of (5).

Let $u_0 = 0 \in U$ be a solution of (4) and consider the linear auxiliary problem

$$D_t w - \mathcal{A}_{ij}(0) D_{ij} w = \mathcal{F}(0), \qquad w \in \mathfrak{W}_p(Q) \ p > n+2, \tag{7}$$

which is uniquely solvable according to $\mathbf{H_3}$). Let U_0 and W_0 be two neighbourhoods of zero such that the inclusion $\{u + w : (u,w) \in U_0 \times W_0\} \subset U$ holds true. Now, we look for solutions $(u,w) \in (U_0 \cap \mathfrak{W}_p(Q)) \times W_0$ of the nonlinear problem

$$D_t(u+w) - \mathcal{A}_{ij}(u+w) D_{ij}(u+w) = \mathcal{F}(u+w). \tag{8}$$

Define the operators

$$\begin{aligned} \mathcal{A}'_{ij}(u,w) =& \mathcal{A}_{ij}(u+w) = a^{ij}(x,t,u+w,D(u+w)) \\ \mathcal{F}'(u,w) =& \mathcal{F}(u+w) - \mathcal{F}(0) + (\mathcal{A}_{ij}(u+w) - \mathcal{A}_{ij}(0)) D_{ij} w \end{aligned}$$

which because of hypotheses $\mathbf{H_1}$) and Lemma 2 are C^1-maps

$$\begin{aligned} &\mathcal{A}'_{ij}(u,w) \in C^1((U_0 \cap \mathfrak{W}_p(Q)) \times W_0; L^\infty(Q)) \\ &\mathcal{F}'(u,w) \in C^1((U_0 \cap \mathfrak{W}_p(Q)) \times W_0; L^p(Q)). \end{aligned}$$

Then, making use of (7), we rewrite (8) in the form

$$D_t u - \mathcal{A}'_{ij}(u,w) D_{ij} u = \mathcal{F}'(u,w), \quad u \in \mathfrak{W}_p(Q), \ p > n+2. \tag{9}$$

Since $\mathcal{A}'_{ij}(0,0) = \mathcal{A}_{ij}(0)$, $\mathcal{F}'(0,0) = 0$ the pair $(u,w) = (0,0) \in U_0 \times W_0$ is a solution of (9). The following result gives a smooth dependence of the solution of (4) from the data.

Theorem 1. *Let U_0 and W_0 be as above. Then there exist neighbourhoods $U_1 \subset U_0$ and $W_1 \subset W_0$ of zero and a solution map $\Phi : C^1(W_1; \mathfrak{W}_p(Q))$ such that the pair $(u, w) \in U_1 \times W_1$ is a solution of (9) if and only if $u = \Phi(w)$.*

One cannot expect that the solution to problem (4) exists on arbitrarily long time interval $\mathcal{S} = (0, T)$ without imposing additional structural or growth conditions on the data. Setting $\mathcal{S}_\tau = (0, \tau)$, $Q_\tau = \mathcal{S}_\tau \times \Omega$ and $U_\tau = \{u|_{Q_\tau} : u \in U\}$, our next assertion deals with local in time solutions of (4).

Theorem 2. *Suppose conditions $\mathbf{H_1}) - \mathbf{H_4})$ hold true and $0 \in U_\tau$ for some $\tau \in \mathcal{S}$. Then there exists at least one solution $u_\tau \in U_\tau \cap \mathfrak{W}_p(Q_\tau)$ to (4).*

The next result gives uniqueness of that solution.

Theorem 3. *Let $\mathbf{H_1}) - \mathbf{H_4})$ hold true and suppose $u, v \in U \cap \mathfrak{W}_p(Q)$ be two solutions of (4), then $u \equiv v$.*

Theorem 4 (Main result)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, $\tau \in (0, T)$ be as in Theorem 2, $u_{0\tau}$ be a local in time solution of (4) under the hypothesis $\mathbf{H_1}) - \mathbf{H_4})$ and $K \subset D$ be a compact such that $(u_{0\tau}, Du_{0\tau}) \in K$ for a.a. $(x, t) \in Q_\tau$. Then there exist neighbourhoods $V_\tau \subseteq \mathfrak{C}^1(Q_\tau \times \bar{D})^{n^2} \times \mathfrak{C}^1(Q_\tau \times \bar{D})$ of $(0, 0)$, $W_\tau \subseteq U_\tau \cap \mathfrak{W}_p(Q_\tau)$ of $u_{0\tau}$ and a C^1-map $\Phi : V_\tau \to W_\tau$ with $\Phi(0, 0) = u_{0\tau}$, such that for all $\big(\{\tilde{a}^{ij}\}_{ij=1}^n, \tilde{f}\big) \in V_\tau$ and $u_\tau \in W_\tau$ holds*

$$\begin{cases} D_t u_\tau - \big(a^{ij}(x, t, u_\tau, Du_\tau) - \tilde{a}^{ij}(x, t, u_\tau, Du_\tau)\big) D_{ij} u_\tau & \\ \qquad = f(x, t, u_\tau, Du_\tau) - \tilde{f}(x, t, u_\tau, Du_\tau) & a.a.\ (x, t) \in Q_\tau \\ u_\tau = 0 & on\ \partial Q_\tau, \end{cases}$$

if and only if $u = \Phi\big(\{\tilde{a}^{ij}\}_{ij=1}^n, \tilde{f}\big)$.

Proof. Let $\tilde{\mathbf{a}} = \{\tilde{a}^{ij}\}_{ij=1}^n \in \mathfrak{C}^1(Q_\tau \times \bar{D})^{n^2}$ be the perturbing matrix and

$$\mathcal{U}_\tau = \{u_\tau \in U_\tau \cap \mathfrak{W}_p(Q_\tau) :\ (u_\tau, Du_\tau) \in K \subset D\}$$

be an open set in $\mathfrak{W}_p(Q_\tau)$. Define

$$A_{ij}(a + \tilde{a}; u_\tau) = a^{ij} + \tilde{a}^{ij}, \quad F(f + \tilde{f}; u_\tau) = f + \tilde{f}$$

and by $\mathbf{H_1})$ and Lemma 2 we have

$$(10) \qquad \begin{cases} \tilde{\mathcal{P}}(\tilde{a}, \tilde{f}, u_\tau) = D_t u_\tau - A_{ij}(a + \tilde{a}; u_\tau) D_{ij} u_\tau \\ \qquad\qquad -F(f + \tilde{f}; u_\tau) = 0 \\ \tilde{\mathcal{P}} \in C^1\big(\mathfrak{C}^1(Q_\tau \times \bar{D})^{n^2} \times \mathfrak{C}^1(Q_\tau \times \bar{D}) \times \mathcal{U}_\tau; L^p(Q_\tau)\big), \end{cases}$$

where $\tilde{\mathcal{P}}(0,0,u_{0\tau}) = D_t u_{0\tau} - \mathcal{A}_{ij}(u_{0\tau}) D_{ij} u_{0\tau} - \mathcal{F}(u_{0\tau}) = 0$. We are going to resolve (10) with respect to u_τ nearby the solution $(0,0,u_{0\tau})$ by means of the IFT. For this goal we need to show that the derivative operator

$$D_u\tilde{\mathcal{P}}(0,0,u_{0\tau})v_\tau = D_t v_\tau - A_{ij}(a;u_{0\tau})D_{ij}v_\tau - D_u F(a;u_{0\tau})v_\tau - D_u A_{ij}(a;u_{0\tau})D_{ij}u_{0\tau}v_\tau$$

is an isomorphism. It is a sum of two linear operators

$$\begin{aligned} v_\tau &\to D_t v_\tau - A_{ij}(a;u_{0\tau})D_{ij}v_\tau : \quad U_\tau \cap \mathfrak{W}_p(Q_\tau) \to L^p(Q_\tau) \\ v_\tau &\to D_u F(f;u_{0\tau})v_\tau + D_u A_{ij}(a;u_{0\tau})D_{ij}u_{0\tau}v_\tau : \ U_\tau \cap \mathfrak{W}_p(Q_\tau) \to L^p(Q_\tau). \end{aligned}$$

The first one is an isomorphism while the second one is the compact operator

$$D_u\mathcal{A}_{ij}(u_{0\tau})D_{ij}u_{0\tau}v_\tau + D_u\mathcal{F}(u_{0\tau})v_\tau$$

because of the compactness of the embedding $W^{2,1}_p \hookrightarrow W^{1,p}_x$. The operator $D_u\tilde{\mathcal{P}}(0,0,u_{0\tau})$ is a Fredholm one (index zero) and it is injective because of $\mathbf{H_4}$), i.e.

$$D_u\tilde{\mathcal{P}}(0,0,u_{0\tau}) \in \mathbf{Iso}(U_\tau \cap \mathfrak{W}_p(Q_\tau); L^p(Q_\tau)).$$

The assertion of the theorem follows by the IFT applied to $\tilde{\mathcal{P}}(\tilde{a},\tilde{f},u_\tau)$. □

3. Newton Iteration Procedure. In order to apply the NIP we consider once again (4) and its linearization (5) along with the following sequence:

$$\text{(11)} \qquad \begin{cases} D_t u_{k+1} - \mathcal{A}_{ij}(u_k)D_{ij}u_{k+1} - D_u\mathcal{A}_{ij}(u_k)D_{ij}(u_{k+1}-u_k) \\ \qquad = \mathcal{F}(u_k) + D_u\mathcal{F}(u_k)(u_{k+1}-u_k) \end{cases}$$

for a given u_k, $k = 1, 2, \dots$.

Theorem 5. *Suppose the conditions* $\mathbf{H_1}) - \mathbf{H_4}$) *hold true, then there exists a neighbourhood* $W \subset U \cap \mathfrak{W}_p(Q)$ *of* u_0 *such that for each* $u_1 \in W$ *there is a unique sequence* $\{u_k\}_{k\in\mathbb{N}} \in W$ *of solutions to* (11) *converging to* u_0 *in* $\mathfrak{W}_p(Q)$, $p > n+2$, *i.e.* $\|u_k - u_0\|_{\mathfrak{W}_p(Q)} \to 0$ *as* $k \to \infty$.

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