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ON THE CONCEPT OF MECHANICAL SYSTEM IN RATIONAL MECHANICS

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Abstract

The paper introduces new notions of vector calculus – homogeneous and inhomogeneous slider-functions and screw-measures. The fundamental principle of dynamics is postulated with their help as well as main scalar and screw measures of rational mechanics and its central concept – *mechanical system*. All classical systems (mass-points, rigid bodies, continua, point-bodies, etc.) are realization of this concept.

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1. Introduction. According to [1] "*Rational Mechanics is the part of mathematics that provides and develops logical models for the enforced changes of place and shape we see everyday things suffer... Mechanics does not study natural things directly. Instead, it considers bodies, which are mathematical concepts designed to abstract some common features of many natural things. One such feature is the mass assigned to each body. Always, a natural body is at any one instant found to occupy a set of places; that set is the shape of that body at that instant ... The change of shape undergone by a body from one instant to another is called the motion of that body ... motions of bodies are conceived as resulting from or at least being invariably accompanied by the action of forces ... Mechanics relates the motions of bodies to the masses assigned to them and the forces that act on them. Bodies are encountered only in their shapes. Masses and forces, therefore, can be correlated with experience in nature only when they are assigned to the shapes of bodies.*"

As a primitive concept we shall define that of mechanical system: the aggregate of shapes (places of body points) with kinematical and dynamic structures attributed by them.

2. Elements of rational mechanics. In what follows, we shall use the set $\mathbf{R}$ of all real numbers and n-dimensional affine space $\mathbf{A}_n$ modelled on n-dimensional vector space $\mathbf{V}_n$.

2.1. Slider-functions and screw measures. According to the Great Soviet Encyclopaedia, v. 5 (Moscow: Soviet Encyclopaedia, 1971), screw calculus is the section of vector calculus in which operations over screws are studied. Here the screw is called a pair of vectors $\{\vec{p}, \vec{q}\}$, bounded at a point O and satisfying to conditions: at transition to a new point O' the vector $\vec{p}$ does not change, and the vector $\vec{q}$ is replaced with a vector $\vec{q}' = \vec{q} - \overrightarrow{OO'} \times \vec{p}$, where $\times$ means cross-product.

Let us detail this notion. Assume that there are vectors $\vec{p}_x$ and $\vec{q}_x$ bounded at a given point $x \in \mathbf{A}_3$, and at any point $y \in \mathbf{A}_3$

$$\vec{p}_y \overset{def}{=} \vec{p}_x, \quad \vec{q}_y \overset{def}{=} \vec{q}_x + r^{\times}_{yx}\vec{p}_x, \tag{1}$$

where $r^{\times}_{yx}$ is the spin-tensor generated by the vector $\vec{r}_{yx} = \overrightarrow{yx}$.

Definition 1. *The field* $l^{p_x,q_x} = \{\vec{p}_y, \vec{q}_y, \forall y \in \mathbf{A}_3\}$ *is called* slider-function *or, briefly,* slider *while* $l^{p_x,q_x}_y \overset{def}{=} \{\vec{p}_y, \vec{q}_y\}$ *is known as* reduction *of the slider w.r.t.* the reduction point $y \in \mathbf{A}_3$.

A slider is called *homogeneous* if $\vec{q}_x = 0$. In this case we shall use the notation l^{p_x}.

If one marks coordinate columns of vectors in a Cartesian frame $\mathcal{E}_0$ with the superscript ^{0}, then $l^{p_x,q_x,0}=\{p^0_y, q^0_y, \forall y \in \mathbf{A}_3\}$ is the coordinate representation of the slider l^{p_x,q_x}. In order to apply the matrix tools one may use the following coordinate columns $l^{p_x,q_x,wr,0}_y = \mathrm{col}\{p^0_y, q^0_y\}$ and $l^{p_x,q_x,tw,0}_y = \mathrm{col}\{q^0_y, p^0_y\}$ known as *wrench* and *twist*, respectively.

Let σ_3 be σ-algebra of subsets in $\mathbf{A}_3$. Introduce the following Borel measure

$$\mu(A) = \mu_{ac}(A) + \mu_{pp}(A), \quad A \in \sigma_3,$$

where $\mu_{ac}(A)$ is the absolutely continuous component w.r.t. Lebesgue measure μ_3 and $\mu_{pp}(A)$ is the pure point (discrete) component presented as $\mu_{pp}(A) = \sum_k \mu_k$ for points $x_k \in A$ which are called *pure*, the others being called *continuous* [2].

Definition 2. *Let* χ_A *be the* characteristic function *of* A. *The Lebesgue–Stieltjes integral*

$$\pi(A) = \int \chi_A l^{p_x,q_x} \mu(dx), \quad A \in \sigma_3 \tag{2}$$

is called signed[1] screw measure *or, briefly,* screw[2,3].

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[1] It is a generalization of the notion of measure by allowing it to have negative values [3].

[2] The screw measure is a screw in the sense of the Encyclopaedia definition.

[3] We shall use this name for surface Lebesgue–Stieltjes integrals, too. Screws generated by homogeneous (inhomogeneous) sliders will be called homogeneous (inhomogeneous).

Introduce a Cartesian frame $\mathcal{E}_p$ and the rotation matrix $C_{0,p}$ defining orientation of the frame $\mathcal{E}_p$ w.r.t $\mathcal{E}_0$ (here for any free vector $\vec{\lambda}$ there is the following relation $\lambda^0 = C_{0,p}\lambda^p$). For $\vec{d}_{0,p} = \overrightarrow{(O_0, O_p)}$ let us define the skew matrices $d_{0,p}^{0\times}$ and $d_{0,p}^{p\times}$.

Theorem 1 ([5]). *A given screw π, $\pi_0^{wr,0} = L_{0,p}^{wr}\pi_p^{wr,p}$, where $\pi_0^{wr,0}$ and $\pi_p^{wr,p}$ are wrenches of π computed in the frames $\mathcal{E}_0$ and $\mathcal{E}_p$, respectively*[4], *the matrix $L_{0,p}^{wr}$ has the representation*

$$
\begin{aligned}
L_{0,p}^{wr} &= D_{0,p}^{0} C_{0,p}^{\otimes} = C_{0,p}^{\otimes} D_{0,p}^{p}, \quad C_{0,p}^{\otimes} = \begin{bmatrix} C_{0,p} & O \\ O & C_{0,p} \end{bmatrix}, \\
D_{0,p}^{0} &= \begin{bmatrix} I & O \\ d_{0,p}^{0\times} & I \end{bmatrix}, \quad D_{0,p}^{p} = \begin{bmatrix} I & O \\ d_{0,p}^{p\times} & I \end{bmatrix},
\end{aligned} \tag{3}
$$

I is the identity matrix, O is the zero one.

2.2. Slider tensor-functions. The slider notion is based on the pair of vector-functions $\vec{p}_x$ and $\vec{q}_x$. That is why these sliders are called *vector* ones. If we replace these vector-functions with tensors $\mathcal{P}_x$ and $\mathcal{Q}_x$ of II rank, then the corresponding sliders will be called *tensor* ones.

2.3. Main concepts and structures of mechanics[5]. In what follows, we shall use *Galilean space-time* [6] introduced as the quadruple $\mathbf{G} = \{\mathbf{V}_4, \mathbf{A}_4, \tau, g\}$, where

1. τ: $\mathbf{V}_4 \to \mathbf{V}_1$ *is a surjective linear mapping called* time *one,*
2. $g = \langle\cdot,\cdot\rangle$ *is an inner product on* ker τ $(= \mathbf{V}_3)$.

The points of $\mathbf{A}_4$ are called *world points* or *events*. The number $\tau(b-a)$ is called *time interval* between events a and $b \in \mathbf{A}_4$. These events a and $b \in \mathbf{A}_4$ are called *simultaneous* if $\tau(a-b) = 0$. The set of simultaneous events forms 3-dimensional affine space $\mathbf{A}_3 \subset \mathbf{A}_4$.

The inner product $\langle\cdot,\cdot\rangle$ (in Galilean space-time) enables one to pass from the space $\mathbf{V}_3$ to *Euclidean* space $\mathbf{E}_3$ with the norm $\|\vec{x}\| = \sqrt{\langle\vec{x},\vec{x}\rangle}$ and to introduce Cartesian frame $\mathcal{E}_0$ in $\mathbf{A}_3$ (with the origin $O_0 \in \mathbf{A}_3$).

Any set $\mathbf{T} \subset \mathbf{R}$ may be used for parameterization of the image of τ with σ-algebra σ_t of subsets in $\mathbf{R}$. Values of parameter $t \in \mathbf{T}$ are called *instants*. We shall assume that there is defined σ-algebra σ_t and the Lebesgue measure $\mu(dt)$ on the set $\mathbf{T}$.

Let us introduce the following notions [1]. *World-line* is a curve in $\mathbf{A}_4$, whose image in $\mathbf{A}_3 \times \mathbf{T}$ associates one point $x(t) \in \mathbf{A}_3$ to each instant $t \in \mathbf{T}$. A collection of non-intersectional world-lines forms *world-tube*. Henceforth, we shall

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[4] Here $\pi_0 \stackrel{def}{=} \pi_y$ for $y = O_0$ and $\pi_p \stackrel{def}{=} \pi_y$ for $y = O_p$.

[5] For the sake of brevity, we do not consider the thermodynamics [1].

name some world-tube $\tilde{\Lambda} \subset \mathbf{A}_4$ as *universe*. A given world-tube $\Lambda \subset \tilde{\Lambda}$, the world-tube $\Lambda^e = \tilde{\Lambda} \setminus \Lambda$ is called *environment* of Λ in the universe.

The universe $\tilde{\Lambda}$ defines the family $\{\tilde{\Lambda}_t \subset \mathbf{A}_3,\ t \in \mathbf{T}\}$, for any world-tube $\Lambda \subset \tilde{\Lambda}$ we have the family $\{\Lambda_t \subset \tilde{\Lambda}_t,\ t \in \mathbf{T}\}$. We shall assume that the Borel measure introduced above is time-invariant on the sets Λ_t and, if a point $x(t_*)$ of any curve $\{x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}\}$ is pure (or continuous) at some time instant t_*, all points of this curve are also pure (or continuous).

For each point $x(t)$ of $\{x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}\}$, the radius-vector $\vec{r}_x(t) = \overrightarrow{(O_0, x(t))}$ is called *position* of the point, and a vector $\vec{v}_x = \vec{v}(x(t), t) \overset{def}{=} \vec{r}_x^{\,\bullet}(t)$ is called its *velocity*[6] w.r.t. O_0 at instant $t \in \mathbf{T}$.

We shall call *mass* the measure $\mathcal{M}(\Lambda_t)$, being continuois w.r.t. $\mu(dx)$. According to Radon–Nicodym theorem the measure may be represented as the following Lebesgue–Stieltjes integral:

$$\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x \mu(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t,$$

where ρ_x is the mass density.

Let us define the following scalar measure:

$$\mathcal{K}(\Lambda_t) = \int \chi_{\Lambda_t} k_x \mu(dx) \tag{4}$$

with the density $k_x = \frac{1}{2}\langle \vec{v}_x, \rho_x \vec{v}_x \rangle$.

Introduce the following vector and screw:

$$\vec{p}_x = \frac{\partial}{\partial \vec{v}_x} k_x = \rho_x \vec{v}_x, \quad \mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t} l^{p_x} \mu(dx).$$

We shall use the notion of *bi-measure* [5]: a vector-function $\Phi(\cdot,\cdot)$, which is defined on $\sigma_3 \times \sigma_3$ and a screw measure of the kind (2) by each of the arguments, is called *screw bi-measure*. A bi-measure $\Phi(A, B)$ is called skew if $\Phi(A, B) = -\Phi(B, A)$ for any A and $B \in \sigma_3$.

Let a skew screw bi-measure $\Phi(\Lambda_t, \Lambda_t^e)$ be homogeneous. By definition it is the screw w.r.t. every argument. That is why there exists the slider l^{f_x} such that the bi-measure coincides (by the first argument) with the following screw:

$$\mathcal{F}(\Lambda_t) = \int \chi_{\Lambda_t} l^{f_x} \mu(dx) \overset{def}{=} \Phi(\Lambda_t, \Lambda_t^e) \tag{5}$$

and $\Phi(\Lambda_t^e, \Lambda_t) = -\mathcal{F}(\Lambda_t)$.

The following proposition represents the essence of dynamics (see also [4,7–9]).

Fundamental principle of dynamics. *For a mechanical tube $\Lambda \subset \tilde{\Lambda}$ there exist a Cartesian frame $\mathcal{E}_0$ and a parameterization $\mathbf{T}$ of the τ-image such that the vector fields $\vec{r}_x$ and $\vec{v}_x$ are solutions of the following equation*:

[6] To honor Newton, we use the superscript $^\bullet$ for full derivatives by t.

$$(6)\qquad \frac{d}{dt}\mathcal{P}^0(\Lambda_t) = \mathcal{F}^0(\Lambda_t), \quad \Lambda_t \subset \tilde{\Lambda}_t,\ t \in \mathbf{T}.$$

In this case

1. *the frame* $\mathcal{E}_0$ *and the parameterization* $\mathbf{T}$ *are called* inertial[7];
2. *the aggregate* $\alpha = \{\sigma_3,\ \sigma_t,\ \mu,\ \forall t \in \mathbf{T},\ \Lambda_t \subset \tilde{\Lambda}_t,\ \mathcal{P}(\Lambda_t),\ \mathcal{F}(\Lambda_t)\}$ *is called* mechanical system;
3. *the set* Λ_t *is called* (actual) shape *undergone by the mechanical system at* $t \in \mathbf{T}$;
4. *the differentiable map* $\mathbf{T} \to \{\Lambda_t,\ t \in \mathbf{T}\}$ *is called* motion *of the mechanical system* [6];
5. *relation* (6) *is called* motion equation[8];
6. *the integral* $\mathcal{K}(\Lambda_t)$ *is called* scalar measure of motion *of the mechanical system*;
7. *the screw* $\mathcal{P}(\Lambda_t)$ *is called* vector measure of motion *of the mechanical system*;
8. *the screw* $\mathcal{F}(\Lambda_t)$ *is called* vector measure of impressed action *of the mechanical system* $\alpha^e = \{\sigma_3,\ \sigma_t,\ \mu,\ \forall t \in \mathbf{T},\ \Lambda_t^e \subset \tilde{\Lambda}_t,\ \mathcal{P}(\Lambda_t^e),\ -\mathcal{F}(\Lambda_t)\}$ *on the mechanical system* α.

2.4. Concept of body. The concept of a body is the subject of various mathematical formalizations. For example, one may represent a body as a pointwise set, an element of Boolean algebra, a differentiable manifold, a topological or measure space [1,5,10], where a map into the space of shapes is considered. But there is a small obstacle: we must also transfer masses and forces to body shares. If we do it in some way, then the mathematical abstraction, body with mass and force, loses the primitive nature. To work out a mathematical theory of mechanics we have all the necessary shares with masses and forces. That is why we shall use the following conventions for a given mechanical system $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \mathcal{P}(\Lambda_t), \mathcal{F}(\Lambda_t)\}$:

1. *the body is that takes some shapes* $\Lambda_t \subset \tilde{\Lambda}_t$ *in 3-dimensional affine space at some instants of time* (*cf. Aristotel, Physics, III,* 5, 204*b*);
2. *the change of shape undergone by a body from one instant to another is called the motion of that body* (*due to the principle of determinacy*);
3. *the positive number* $\mathcal{M}(\Lambda_t)$ *is the body mass*;
4. *the screw* $\mathcal{F}(\Lambda_t)$ *is the force impressed at the body.*

2.5. Generalization of mechanical system concept. The non-trivial nature of the mechanical system concept can be seen from the fact that we may postulate the following equation of motion:

$$(7)\qquad \frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t) + \mathcal{F}_i(\Lambda_t) + \mathcal{F}_c(\Lambda_t),$$

[7] The frame is also called that of *reference*.

[8] Relation (6) can be transformed in the vector form $\frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t)$.

where

$$\mathcal{P}(\Lambda_t) \stackrel{def}{=} \int \chi_{\Lambda_t} l^{p_x,q_x} \mu(dx) \tag{8}$$

is the inhomogeneous *screw measure of motion* of the mechanical system[9]; $\mathcal{F}$ is the inhomogeneous screw measure; the inhomogeneous screw measure $\mathcal{F}_i$ is the so-called *increment velocity* of the measure $\mathcal{P}$; the inhomogeneous screw measure $\mathcal{F}_c$ is the so-called *constraint action.*

We shall assume that $\mathcal{F}_c = \mathcal{F}_{int} + \mathcal{F}_{ext}$, where $\mathcal{F}_{int}$ is formed by internal constraints of the set Λ_t while $\mathcal{F}_{ext}$ is formed by external constraints.

2.6. Derivatives of some measures. Let us consider the set Λ_t as the union of the set Λ_t^{pp} of the pure points entering into it, with the set Λ_t^{ac} of its continuous points. We will assume that the last set has the surface $\partial\Lambda_t^{ac}$, which is Lyapunov's simple closed one [11]. According to Gauss–Ostrogradsky (divergence) theorem we have (see also [10])

$$\frac{d}{dt}\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t^{ac}} (\frac{d}{dt}\rho_x + \rho_x \mathrm{div}\, \vec{v}_x)\mu_{ac}(dx) + \sum_k (\frac{d}{dt}\rho_{x_k})\mu_{pp}(x_k).$$

We shall assume that the function ρ_x is defined by the *continuity equation* for continuous points $\frac{\partial}{\partial t}\rho_x + \mathrm{div}\vec{p}_x = \nu_x$ and for pure points – $\frac{d}{dt}\rho_{x_k} = \nu_{x_k}$, where ν_x depicts the generation (negative in the case of removal) per unit volume and unit time of the measure $\mathcal{M}$. Terms that generate ($\nu_x > 0$) or remove ($\nu_x < 0$) are referred to as 'sources' and 'sinks' respectively.

In what follows, we shall assume that all sliders are homogeneous.

According to [10]

$$\frac{d}{dt}\mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t^{ac}} (\frac{d}{dt} l^{p_x} + l^{p_x} \mathrm{div}\, \vec{v}_x)\mu_{ac}(dx) + \sum_k (\frac{d}{dt} l^{p_{x_k}})\mu_{pp}(x_k). \tag{9}$$

As for continuous points (see also [10]) $\frac{d}{dt}l^{p_x} + l^{p_x}\mathrm{div}\,\vec{v}_x = \rho_x \frac{d}{dt}\, l^{v_x} + \nu_x l^{v_x}$, from (7) and (9) follows:

$$\int \chi_{\Lambda_t} (\rho_x \frac{d}{dt} l^{v_x} + \nu_x l^{v_x})\mu(dx) = \mathcal{F}(\Lambda_t) + \mathcal{F}_i(\Lambda_t) + \mathcal{F}_c(\Lambda_t). \tag{10}$$

3. Specifying mechanical systems. Show how the given above axiomatics relates to the conventional mechanics.

In the first place exemplify the notion of skew screw bi-measure. In the conventional mechanics it is considered that there is the gravitational interaction between bodies. It can be formalized in the following way. Let a skew screw bi-measure $\Psi(\Lambda_t, \Lambda_t^e)$ be such that

[9] It is not necessary to think that $\vec{p} = \rho_x \vec{v}_x$ in (8).

$$\Psi(\Lambda_t, \Lambda_t^e) = \int \chi_{\Lambda_t} l^{g_x} \rho_x \mu(dx), \quad \vec{g}_x = \gamma \int \chi_{\Lambda_t^e} \overrightarrow{(x-y)} \frac{\rho_y \mu(dy)}{\|\overrightarrow{(x-y)}\|^3},$$

where γ is a positive (gravitational) constant.

Then the screw $\mathcal{G}(\Lambda_t) \stackrel{\text{def}}{=} \Psi(\Lambda_t, \Lambda_t^e)$ can be called *measure of gravitating action* of α^e upon α [5]. One may take this screw as the screw measure $\mathcal{F}(\Lambda_t)$ of impressed action.

Assume that the increment velocity of $\mathcal{P}$ is given by the following Lebesgue–Stieltjes integral:

$$\mathcal{F}_i(\Lambda_t) = \int \chi_{\Lambda_t} l^{\xi_x} \mu(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \tag{11}$$

where l^{ξ_x} is its density.

Let no external constraint be.

3.1. A mass-point. Consider a world-line $\Lambda \subset \tilde{\Lambda}$ whose image in $\mathbf{A}_3 \times \mathbf{T}$ generates the curve $\{x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}\}$. Assume that the points $x(t)$ are pure, *i.e.*, $x(t) = x_k(t)$. Then the mechanical system $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, x(t) \in \tilde{\Lambda}_t, \rho_x, \nu_x, \vec{f}_x, \vec{\xi}_x\}$ is called *mass-point.*

From relation (10) follows that:

$$\rho_x \frac{d}{dt} \vec{v}_x + \nu_x \vec{v}_x = \vec{f}_x + \vec{\xi}_x. \tag{12}$$

If $\nu_x \equiv 0$ and $\vec{\xi}_x \equiv 0$, then equation (12) is known as *second Newton's law*, where $\vec{f}_x$ is the impressed force acting at the point $x = x_k \in \tilde{\Lambda}_t$ with the mass $\mathcal{M}_k = \rho_x \mu_k$.

If $\nu_x \neq 0$ and $\vec{\xi}_x = \nu_x \vec{u}_x$, where $\vec{u}_x$ is the velocity of mass gain or loss, then equation (12) is known as that of Meshchersky [9].

3.2. A rigid body. The mechanical system $\alpha_p = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \forall x \in \Lambda_t, \rho_x, \nu_x, \vec{f}_x, \vec{\xi}_x\}$ is called *rigid body* if: 1) the sets Λ_t are bounded and closed; 2) the constraints applied on its points[10] keep distances between them not changing with time; 3) the constraints are *ideal* [12].

As the inner constraints are ideal, from relations (10)–(11) follows that [10]:

$$\rho_x \frac{d}{dt} l_0^{v_x,0} + \nu_x l_0^{v_x,0} = l_0^{f_x,0} + l_0^{\xi_x,0}. \tag{13}$$

At any time instant t^*, consider the set Λ_{t^*}. Let us attach a Cartesian frame $\mathcal{E}_p$ to the set under consideration. It is plain that the frame takes the same position in all sets Λ_t. In the frame these sets are immobile, coincide with one another and form the set noted as Λ_p. We shall say that the frame $\mathcal{E}_p$ is attached to the rigid body α_p.

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[10] A rigid body may contain continuous and pure points [9].

Let $\vec{r}_{x,p}$ be the radius-vector of a point x bounded at the point O_p. According to Euler equation the translation $\vec{v}_{0,p}$ and angular $\vec{\omega}_{0,p}$ velocities generate *kinematical slider* $V_{0,p} = \{\vec{\omega}_{0,p}, \vec{v}_{0,p} + \vec{r}_{x,p} \times \vec{\omega}_{0,p}, \forall x \in \mathbf{A}_3\}$, which causes the twist $V_{0,p}^{tw,p} = \mathrm{col}\{v_{0,p}^p, \omega_{0,p}^p\}$ being known as *quasi-velocity vector*. There is the following representation [5]:

$$l_p^{v_x,wr,p} = \Theta_p^x V_{0,p}^{tw,p}, \quad \Theta_p^x = \begin{bmatrix} I & -r_p^{p\times} \\ r_p^{p\times} & -(r_p^{p\times})^2 \end{bmatrix}.$$

From equation (13) we have

$$\int \chi_{\Lambda_p} [\rho_x L_{0,p}^{wr,-1} \frac{d}{dt} (L_{0,p}^{wr} \Theta_p^x V_{0,p}^{tw,p}) + \nu_x \Theta_p^x V_{0,p}^{tw,p}] \mu(dx) = \int \chi_{\Lambda_p} (l_p^{f_x,p} + l_p^{\xi_x,p}) \mu(dx).$$

As the twist $V_{0,p}^{tw,p}$ does not depend on points x, we may obtain the well-known Newton–Euler equation in quasi-velocities and Lagrange ones in generalized coordinates and velocities [5].

3.3. A continuum. Let $\vec{f}_x = \rho_x \vec{g}_x$. Assume that all points of the sets $\Lambda_t \subset \tilde{\Lambda}_t$ are continuous, μ_2 is the restriction of μ on the surface $\partial\Lambda_t$, $\vec{n}_x$ is the normal to this surface.

According to [13], internal constraints applied on points $x \in \Lambda_t$ cause *stress*. Define the internal constraint action as follows [1, 10]:

$$\mathcal{F}_{int}(\Lambda_t) = \int \chi_{\partial\Lambda_t} l^{\mathcal{T}_x n_x} \mu_2(dx) = \int \chi_{\Lambda_t} \mathrm{div}\, l^{\mathcal{T}_x} \mu(dx),$$

where $\mathcal{T}_x$ is called *stress tensor*.

Take a point $y(t)$ in a small vicinity of $x(t) \in \Lambda_t$ at an instant $t \in \mathbf{T}$ and define their radius-vectors $\vec{r}_x(t)$ and $\vec{r}_y(t)$ (in $\mathcal{E}_0$) and the vector $\vec{h}(t) = \vec{r}_y - \vec{r}_x(t)$. Then there is the Cauchy–Helmholtz relation [10]

$$\vec{v}_y(t) \cong \vec{v}_x(t) + \frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T]\, \vec{h}(t) + \frac{1}{2}[d\vec{v}_x/d\vec{r}_x - (d\vec{v}_x/d\vec{r}_x)^T]\, \vec{h}(t),$$

where $\frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T]$ is known as *tensor of strain velocities*; $\frac{1}{2}[d\vec{v}_x/d\vec{r}_x - (d\vec{v}_x/d\vec{r}_x)^T]$ is known as *spin-tensor* at the point $x \in \tilde{\Lambda}_t$ at the instant t.

Define the tensor $\mathcal{S}_x(t)$ as the solution of the following equation:

$$\dot{\mathcal{S}_x}(t) = \frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T]$$

with initial data $\mathcal{S}_x = \mathcal{I}$, $t = t_0$, $\mathcal{I}$ is the identity (spherical) tensor.

The tensor $\mathcal{S}_x$ is called *strain* one [5]. Let us define $\mathcal{U}_x$ as $\mathcal{S}_x$ or $\dot{\mathcal{S}_x}$.

Definition 3. *The mechanical system* $\alpha = \{\sigma_3,\ \sigma_t,\ \mu,\ \forall t \in \mathbf{T},\ \Lambda_t \subset \tilde{\Lambda}_t,\ \forall x \in \Lambda_t,\ \rho_x,\ \nu_x,\ \vec{g}_x,\ \vec{\xi}_x,\ \mathcal{T}_x\}$ *is called* continuous medium *or* continuum of Navier–Stocks–Lame class *if the tensor* $\mathcal{T}_x$ *is an* isotropic *map of* $\mathcal{U}_x$, i.e., *invariant w.r.t. orthogonal transformations.*

3.3.1. Quasi-linear isotropic matrix-functions. For any 3×3–matrix U the aggregate PUQ is an isotropic function of U if the matrices P and Q are proportional to I with scalar coefficients being invariant w.r.t. rotations.

Define the matrices $E_1 = (\operatorname{tr} U)I,\ E_2 = U,\ E_3 = U^T$, where I is the identity matrix.

Theorem 2. *All isotropic quasi-linear* 3×3*–matrix functions of entries of* U *are given by the following relation* [14]:

$$T = r_1 E_1 + r_2 E_2 + r_3 E_3, \tag{14}$$

where r_i are invariant w.r.t. rotations (they can be functions of the time, invariants of U).

Let U be 2×2 matrix. It is easy to see that for 2×2 matrices P and Q the aggregate PUQ is an isotropic map of U if P and Q are of the kind $rI + \widetilde{r}\widetilde{I}$, where the scalar coefficients r and $\widetilde{r}$ are invariant w.r.t. rotations, I is the identity 2×2 matrix, $\widetilde{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Introduce the following matrices $E_1 = (\operatorname{tr} U)I$, $\widetilde{E}_1 = (\operatorname{tr}\widetilde{I}U)\widetilde{I}$, $E_2 = U$, $\widetilde{E}_2 = \widetilde{I}U$, $E_3 = U^T$, $\widetilde{E}_3 = U^T\widetilde{I}$, $E_4 = \widetilde{I}U^T$, $E_5 = U\widetilde{I}$, $E_6 = \widetilde{I}U\widetilde{I}$ and $E_7 = \widetilde{I}U^T\widetilde{I}$. It is easy to see that there are 6 linearly independent matrices, e.g., E_1, $\widetilde{E}_1$, E_2, $\widetilde{E}_2$, E_3 and $\widetilde{E}_3$.

Thus there is the set of isotropic matrix functions of entries of 2×2–matrix U

$$T = r_1 (\operatorname{tr} U)\, I + \widetilde{r}_1 (\operatorname{tr}\widetilde{I}U)\widetilde{I} + r_2 U + r_3 U^T + \widetilde{r}_2 \widetilde{I}U + \widetilde{r}_3 U^T \widetilde{I}, \tag{15}$$

where r_i and $\widetilde{r}_i$ are parameters being invariant w.r.t. $\mathcal{SO}(\mathbf{R}, 2)$.

3.3.2. Symmetry of stress tensor. According to Gauss–Ostrogradsky (divergence) theorem, equation (10) is equivalent to [1,10]

$$\rho_x \frac{d}{dt}\vec{v}_x + \nu_x \vec{v}_x = \rho_x \vec{g}_x + \vec{\xi}_x + \operatorname{div}\mathcal{T}_x, \quad \mathcal{T}_x = \mathcal{T}_x^T. \tag{16}$$

We may introduce the following *constitutive* relations with the help of symmetrizing relations (14) and (15): in the 3-dimensional case – $\mathcal{T}_x = r_0\mathcal{I} + r_1(\operatorname{tr}\mathcal{U}_x)\mathcal{I} + r_2\mathcal{U}_x$ and in the 2-dimensional case – $\mathcal{T}_x = r_0\mathcal{I} + r_1(\operatorname{tr}\mathcal{U}_x)\mathcal{I} + r_2\mathcal{U}_x + r_3(\widetilde{\mathcal{I}}\mathcal{U}_x - \mathcal{U}_x\widetilde{\mathcal{I}})$, where r_i are *rheological coefficients*; the tensor $\widetilde{\mathcal{I}}$ corresponds to the matrix $\widetilde{I}$.

3.4. Systems with inhomogeneous sliders. 3.4.1. Multiphase systems. Equations (6.34) and (7.11), given in [10], can be realized in the form of (7)–(8).

3.4.2. Eulerian mechanics. From the physical point of view, a mass-point is a body of small dimensions. The body motion is characterised with translational and angular velocities as well as mass and inertia tensor. That is why *Eulerian mechanics* supplies points of the sets Λ_t, $t \in \mathbf{T}$, with translation velocities $\vec{v}_x$ and angular ones $\vec{\omega}_x \in \mathbf{V}_3$, as well as densities A_x, B_x and C_x of generalized inertia tensors [11]. Then the kinetic energy (4) is introduced by its positive defined density $k_x \overset{def}{=} \frac{1}{2}\langle \vec{v}_x, A_x\vec{v}_x\rangle + \langle \vec{v}_x, B_x\vec{\omega}_x\rangle + \frac{1}{2}\langle \vec{\omega}_x, C_x\vec{\omega}_x\rangle$. After that one defines the vectors $\vec{p}_x \overset{def}{=} \frac{\partial}{\partial \vec{v}_x} k_x = A_x\vec{v}_x + B_x\vec{\omega}_x$ and $\vec{q}_y \overset{def}{=} \vec{r}_{yx} \times \vec{p}_x + \vec{q}_x$, where $\vec{q}_x = \frac{\partial}{\partial \vec{\omega}_x} k_x = B_x^T\vec{v}_x + C_x\vec{\omega}_x$ is the density of the so-called *dynamical spin.*

With the help of $\vec{p}_x$ and $\vec{q}_x$ equations (7)–(8) [11]. Realizations of these equations depict motion of point-bodies and their systems, thin rods and so on [11].

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