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HOW TO HAVE A WRONG BET IN FOOTBALL POOLS?

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Abstract

Consider the set $Q_n = \{0, 1, 2\}^n$ equipped with the usual Hamming distance. Denoted by $T(n)$ the minimal number of spheres of radius n that covers Q_n. The exact values of $T(n)$ are known for $n \leq 8$ and in particular $T(8) = 44$. It is known that for any $n \leq 7$ (see [1]) up to equivalence there exists unique covering of Q_n. In this paper we show that up to equivalence there exist two coverings of Q_8 with 44 elements. Also we improve the best known lower bound $T(9) \geq 66$ by showing that $T(9) \geq 67$. Since there exists a covering of Q_9 with 68 spheres we have $67 \leq T(9) \leq 68$. The inequality $T(9) \geq 67$ implies $T(10) \geq 101$, $T(11) \geq 152$, $T(12) \geq 228$ and $T(13) \geq 342$, thus improving the best known lower bounds for $10 \leq n \leq 13$.

1. Introduction. In football pools one bets over 13 games. For each game he chooses between three possible outcomes – win, draw and loss. In usual football pool the goal is to correctly predict as many games as possible. Finding the minimal number of bets that assure certain number of correctly predicted outcomes is known as the *football pool problem.* For more information the reader is referred to [3].

In the inverse football pool problem the aim is, no matter the outcome of the games, to have a bet that is wrong for all games. To put this into mathematical terms consider the set $Q_n = \{0, 1, 2\}^n$ with the usual Hamming distance. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ we define the Hamming distance $\mathbf{d}(\mathbf{x}, \mathbf{y})$ as

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|.$$

A sphere with centre $\mathbf{x}$ and radius r is defined as the set

$$S(\mathbf{x}, r) = \{\mathbf{y} \mid \mathbf{d}(\mathbf{x}, \mathbf{y}) = r\}.$$

A subset A of Q_n is called *covering* if the spheres of radius n centred at the elements of A cover Q_n. In other words $A \subset Q_n$ is a covering if for any $\mathbf{y} \in Q_n$ there exists $\mathbf{x} \in A$ such that $\mathbf{d}(\mathbf{x}, \mathbf{y}) = n$. The minimal cardinality of a covering of

Q_n is denoted by $T(n)$. Finally, a covering A of Q_n is called *optimal* if $|A| = T(n)$. The problem of finding $T(n)$ and all optimal coverings of Q_n is known as the *inverse football pool problem* (see [1]).

The next proposition is straightforward and gives an important recursive bound on $T(n)$.

Proposition 1. *The following inequality holds*

$$T(n) \geq \frac{3}{2}T(n-1).$$

Proof. Consider an optimal covering A of Q_n, i.e. A is covering and $|A| = T(n)$. Partition the elements of A into three parts A_0, A_1 and A_2 according to their last coordinate. We have

$$A = \{\mathbf{x}0 \mid \mathbf{x} \in A_0\} \cup \{\mathbf{x}1 \mid \mathbf{x} \in A_1\} \cup \{\mathbf{x}2 \mid \mathbf{x} \in A_2\}.$$

For $\{i, j, k\} = \{0, 1, 2\}$ it is clear that $A_i \cup A_j$ is a covering of Q^{n-1}. Therefore $A_i + A_j \geq T(n-1)$. The same argument implies that $A_i + A_k \geq T(n-1)$ and $A_j + A_k \geq T(n-1)$. Summing up these inequalities gives

$$2T(n) = 2(C_i + C_j + C_k) \geq 3T(n-1),$$

hence $T(n) \geq \frac{3}{2}T(n-1)$. □

2. Known results. The known results concerning $T(n)$ are given in Table 1 and are taken from [1].

Table 1

n	$T(n)$	n	$T(n)$
1	2	7	29
2	3	8	44
3	5	9	66–68
4	8	10	99–104
5	12	11	149–172
6	18	12	224–264
		13	336–408

The results for $n \leq 6$ and the bounds $T(7) \leq 29$ and $T(8) \leq 44$ are due to the Finish football pool magazine Veikkaaja. It is straightforward to show that $T(1) = 2$ and using the inequality from Proposition 1 we obtain $T(2) \geq 3$, $T(3) \geq 5$, $T(4) \geq 8$, $T(5) \geq 12$, $T(6) \geq 18$. A covering of Q_6 with 18 elements is presented in Table 2 (see [1]).

This implies that $T(2) = 3$, $T(3) = 5$, $T(4) = 8$, $T(5) = 12$, $T(6) = 18$.

Observe that for all $n \leq 6$ we have $T(n) = \left\lceil \frac{3}{2}T(n-1) \right\rceil$.

The first value of n for which $T(n) \neq \left\lceil \frac{3}{2}T(n-1) \right\rceil$ is $n = 7$. It has been shown in [1] by exhaustive computer search that $T(7) = 29$ while the bound from Proposition 1 implies $T(7) \geq 27$. Proposition 1 gives $T(8) \geq 44$ and since a

Table 2

Optimal covering of Q_6

1.	0 0 0 0 0 0	10.	0 2 0 2 1 1
2.	1 1 1 1 0 0	11.	0 1 1 0 2 1
3.	2 2 1 0 1 0	12.	1 0 0 1 2 1
4.	1 0 2 2 1 0	13.	2 2 0 1 0 2
5.	0 2 2 1 2 0	14.	0 1 2 2 0 2
6.	2 1 0 2 2 0	15.	1 1 0 0 1 2
7.	1 2 2 0 0 1	16.	0 0 1 1 1 2
8.	2 0 1 2 0 1	17.	2 0 2 0 2 2
9.	2 1 2 1 1 1	18.	1 2 1 2 2 2

covering of Q_8 with 44 elements exists, we conclude that $T(8) = 44$. The upper bounds for $n = 9$ and $n = 10$ were found in [2] using the so-called *tabu search.*

After having the exact value of $T(n)$, we are interested in finding the number of distinct optimal coverings. Two coverings A and B are equivalent if A is obtained from B by the permutation of the coordinates followed by permutation of the elements of every coordinate. More precisely we have

Definition 1. Two coverings A and B are equivalent if there exists a permutation $\sigma \in S_n$ and n permutations $s_1, s_2, \ldots, s_n$ of $\{0, 1, 2\}$ such that

$$(x_1, x_2, \ldots, x_n) \in A \iff (s_1(x_{\sigma(1)}), s_2(x_{\sigma(2)}), \ldots, s_n(x_{\sigma(n)})) \in B.$$

It is shown in [1] that for any $n \leq 7$ up to equivalence there exists a unique optimal covering of Q_n. For $n = 7$ this involves an exhaustive computer search. The unique optimal covering for $n = 7$ is given in Table 3. Denote this covering by C_7.

Direct verification shows that for any $\mathbf{x}, \mathbf{y} \in C_7$, $\mathbf{x} \neq \mathbf{y}$ we have $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 3$.

Recall that $T(7) = 44$. It is stated in [1] that it is not known whether the optimal covering of Q_8 is unique.

3. Main results. We describe all optimal coverings of Q_8, i.e. all coverings with 44 elements. The result is given in the following theorem.

Table 3

The unique optimal covering C_7 of Q_7

1.	0 0 0 0 0 0 0	11.	2 2 2 2 2 2 0	21.	1 2 1 2 2 2 1
2.	1 1 1 1 0 0 0	12.	2 1 2 1 0 0 1	22.	0 1 1 0 2 0 2
3.	2 2 1 0 1 0 0	13.	0 2 0 2 0 0 1	23.	1 0 0 1 2 0 2
4.	1 0 2 2 1 0 0	14.	1 2 2 0 1 0 1	24.	0 2 2 1 2 1 2
5.	1 2 2 0 0 1 0	15.	2 0 1 2 1 0 1	25.	2 1 0 2 2 1 2
6.	2 0 1 2 0 1 0	16.	2 2 1 0 0 1 1	26.	1 1 0 0 0 2 2
7.	2 1 2 1 1 1 0	17.	1 0 2 2 0 1 1	27.	0 0 1 1 0 2 2
8.	0 2 0 2 1 1 0	18.	0 0 0 0 1 1 1	28.	2 2 0 1 1 2 2
9.	1 0 1 0 2 2 0	19.	1 1 1 1 1 1 1	29.	0 1 2 2 1 2 2
10.	0 1 0 1 2 2 0	20.	2 0 2 0 2 2 1		

Theorem 1. *Up to equivalence, there exist two optimal coverings of Q_8.*

The properties of two coverings of Q_8 make it possible to improve the best known lower bound for $T(9) \geq 66$.

Theorem 2. *It is true that $T(9) \geq 67$.*

Combining the result of Theorem 2 and the inequality $T(n) \geq \frac{3}{2}T(n-1)$ we improve all lower bounds for $T(n)$ when $10 \leq n \leq 13$. We have $T(10) \geq 101$, $T(11) \geq 152$, $T(12) \geq 228$ and $T(13) \geq 342$.

4. Proof of Theorem 1. Consider an optimal covering A of Q_8, i.e. $|A| = 44$.

For fixed coordinate position partition the elements of A into three sets A_0, A_1 and A_2 according to the entry in this position. Without loss of generality assume that $|A_1| = \min\{|A_0|, |A_1|, |A_2|\}$. Further, we have that $|A_0| + |A_1| + |A_2| = 44$ and since $T(7) = 29$ we obtain that $|A_i| + |A_j| \geq 29$ for any i, j, $i \neq j$, $i, j \in \{0, 1, 2\}$. It follows that $|A_1| = 14$ and $|A_0| = |A_2| = 15$.

Observe that $|A_0| + |A_1| = |A_1| + |A_2| = 29$. Therefore $A_0 \cup A_1$ and $A_1 \cup A_2$ are both equivalent to C_7. Hence any of A_0, A_1 and A_2 is equivalent to a subset of C_7.

In other words for any $i \in \{0, 1, 2\}$ and for any coordinate position all elements in A having i in this position are equivalent to a subset of C_7.

Proposition 2. *If A is an optimal covering of Q_8, then for any $\mathbf{u}, \mathbf{v} \in A$ we have $\mathbf{d}(\mathbf{u}, \mathbf{v}) \geq 3$.*

Proof. Suppose $\mathbf{u}, \mathbf{v} \in A$ where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are such that $\mathbf{d}(\mathbf{u}, \mathbf{v}) < 3$. Without loss of generality assume that $u_n = v_n$ and then $\mathbf{d}((u_1, u_2, \ldots, u_{n-1}), (v_1, v_2, \ldots, v_{n-1})) < 3$. By the above observation $(u_1, u_2, \ldots, u_{n-1})$ and $(v_1, v_2, \ldots, v_{n-1})$ can be considered as elements of C_7. But we know that $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 3$ for any $\mathbf{x}, \mathbf{y} \in C_7$, a contradiction. $\square$

Consider a partition of A into A_0, A_1 and A_2 according to last coordinate entry. Without loss of generality assume that $A_1 \cup A_2 = C_7$. We know that $A_1 \cup A_0$ is equivalent to C_7. According to Proposition 1 we have $A_0 \cap A_1 = \varnothing$ (otherwise there exist $\mathbf{u}$ and $\mathbf{v}$ from A such that $\mathbf{d}(\mathbf{u}, \mathbf{v}) = 1$) which implies that $|(A_1 \cup A_0) \cap (A_1 \cup A_2)| = |A_1| = 14$. Therefore A_1 is the intersection of two copies of C_7.

Since there are 7! coordinate permutations and for each coordinate there are $3! = 6$ permutations of the elements $0, 1, 2$ we have $7!.6^7$ copies of given covering. Some of these copies coincide depending on the full automorphism group of the given covering. It is easily checked that the full automorphism group of C_7 has order 48.

Computer search goes through all $7!.6^7$ copies of C_7 and finds all copies C_7' with the property $|C_7' \cap C_7| = 14$. Then the elements of A_1 are exactly the 14 elements from $C_7' \cap C_7$, the elements of A_2 are $C_7 \setminus A_1$ and the elements of A_0 are $C_7' \setminus A_1$.

As a result we obtain 384 sets with 44 elements. For any such set we check whether it is a covering of Q_8 (it only remains to be seen whether $A_0 \cup A_2$ is a covering of Q_7). It turns out that all sets obtained are indeed coverings. Standard check for equivalence shows that up to equivalence there exist only two coverings C_8^1 and C_8^2. The two coverings are given in Table 4 and Table 5.

Table 4

Optimal covering C_8^1 of Q_8

1.	0 0 0 0 0 0 0 2	16.	2 2 1 0 0 1 1 1	31.	2 2 1 2 0 0 2 0
2.	1 1 1 1 0 0 0 1	17.	1 0 2 2 0 1 1 1	32.	0 0 0 2 1 0 2 0
3.	2 2 1 0 1 0 0 2	18.	0 0 0 0 1 1 1 1	33.	0 2 0 0 0 1 2 0
4.	1 0 2 2 1 0 0 2	19.	1 1 1 1 1 1 1 2	34.	2 0 1 0 1 1 2 0
5.	1 2 2 0 0 1 0 2	20.	2 0 2 0 2 2 1 2	35.	1 2 2 2 1 1 2 0
6.	2 0 1 2 0 1 0 2	21.	1 2 1 2 2 2 1 2	36.	1 1 2 1 2 2 2 0
7.	2 1 2 1 1 1 0 1	22.	0 1 1 0 2 0 2 1	37.	2 1 0 0 2 0 1 0
8.	0 2 0 2 1 1 0 2	23.	1 0 0 1 2 0 2 2	38.	0 1 1 2 2 1 1 0
9.	1 0 1 0 2 2 0 1	24.	0 2 2 1 2 1 2 2	39.	2 0 0 1 0 2 1 0
10.	0 1 0 1 2 2 0 2	25.	2 1 0 2 2 1 2 1	40.	0 2 1 1 1 2 1 0
11.	2 2 2 2 2 2 0 1	26.	1 1 0 0 0 2 2 2	41.	0 0 2 1 2 0 0 0
12.	2 1 2 1 0 0 1 2	27.	0 0 1 1 0 2 2 1	42.	1 2 0 1 2 1 0 0
13.	0 2 0 2 0 0 1 1	28.	2 2 0 1 1 2 2 1	43.	0 1 2 0 0 2 0 0
14.	1 2 2 0 1 0 1 1	29.	0 1 2 2 1 2 2 2	44.	1 1 0 2 1 2 0 0
15.	2 0 1 2 1 0 1 1	30.	1 0 2 0 0 0 2 0		

Table 5

Optimal covering C_8^2 of Q_8

1.	0 0 0 0 0 0 0 2	16.	2 2 1 0 0 1 1 2	31.	1 2 1 2 0 0 2 0
2.	1 1 1 1 0 0 0 2	17.	1 0 2 2 0 1 1 2	32.	0 2 0 2 2 2 2 0
3.	2 2 1 0 1 0 0 1	18.	0 0 0 0 1 1 1 1	33.	2 1 2 1 2 2 2 0
4.	1 0 2 2 1 0 0 1	19.	1 1 1 1 1 1 1 1	34.	1 0 1 0 1 1 2 0
5.	1 2 2 0 0 1 0 1	20.	2 0 2 0 2 2 1 1	35.	2 2 2 2 1 1 2 0
6.	2 0 1 2 0 1 0 1	21.	1 2 1 2 2 2 1 1	36.	0 1 0 1 1 1 2 0
7.	2 1 2 1 1 1 0 2	22.	0 1 1 0 2 0 2 1	37.	1 1 0 0 2 0 1 0
8.	0 2 0 2 1 1 0 2	23.	1 0 0 1 2 0 2 1	38.	0 0 1 1 2 0 1 0
9.	1 0 1 0 2 2 0 2	24.	0 2 2 1 2 1 2 2	39.	0 1 1 0 0 2 1 0
10.	0 1 0 1 2 2 0 2	25.	2 1 0 2 2 1 2 2	40.	1 0 0 1 0 2 1 0
11.	2 2 2 2 2 2 0 2	26.	1 1 0 0 0 2 2 1	41.	2 1 0 2 1 2 0 0
12.	2 1 2 1 0 0 1 1	27.	0 0 1 1 0 2 2 1	42.	0 2 2 1 1 2 0 0
13.	0 2 0 2 0 0 1 1	28.	2 2 0 1 1 2 2 2	43.	0 1 2 2 2 1 0 0
14.	1 2 2 0 1 0 1 2	29.	0 1 2 2 1 2 2 2	44.	2 2 0 1 2 1 0 0
15.	2 0 1 2 1 0 1 2	30.	2 0 2 0 0 0 2 0		

Remark. To check that C_8^1 and C_8^2 are not equivalent we count the number of pairs $(\mathbf{u}, \mathbf{v})$ from each covering such that $\mathbf{d}(\mathbf{u}, \mathbf{v}) = t$ for any t, $1 \le t \le 8$. It is clear that the numbers obtained are invariants under equivalence transformations.

We have

t	1	2	3	4	5	6	7	8
Pairs from C_8^1 at distance t	0	0	0	210	320	240	128	48
Pairs from C_8^2 at distance t	0	0	0	222	320	216	128	60

Therefore the two coverings are not equivalent. Note that for any two elements $\mathbf{x}$, $\mathbf{y}$ from C_8^i, $i = 1, 2$, it is true that $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 4$.

5. Proof of Theorem 2. Assume $T(9) = 66$ and consider a covering A of Q_9 with 66 elements. As above, partition the elements of A into three sets A_0, A_1 and A_2. Since for any $i, j \in \{0, 1, 2\}$, $i \neq j$, it is true that $|A_i| + |A_j| \geq T(8) = 44$ and $|A_0| + |A_1| + |A_2| = 66$ we have that

$$|A_0| = |A_1| = |A_2| = 22.$$

Hence for any $i, j \in \{0, 1, 2\}$, $i \neq j$, the set $A_i \cup A_j$ is an optimal covering of Q_8. Therefore for any $i, j \in \{0, 1, 2\}$, $i \neq j$, the set $A_i \cup A_j$ is equivalent to one of the two optimal coverings of Q_8.

Proposition 3. *For any two elements* $\mathbf{u}$ *and* $\mathbf{v}$ *from a covering of* Q_9 *with* 66 *elements we have* $\mathbf{d}(\mathbf{u}, \mathbf{v}) \geq 5$.

Proof. Suppose $\mathbf{u} = (u_1, u_2, \ldots, u_9)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_9)$ are elements of optimal covering and $\mathbf{d}(\mathbf{u}, \mathbf{v}) \leq 4$. Without loss of generality assume $u_9 \neq v_9$. Thus $\mathbf{d}((v_1, v_2, \ldots, v_8), (u_1, u_2, \ldots, u_8)) \leq 3$ which is impossible since the elements $(v_1, v_2, \ldots, v_8)$ and $(u_1, u_2, \ldots, u_8)$ being elements of $A_{u_9} \cup A_{v_9}$ belong to optimal covering of Q_8. $\square$

It is easy to find three elements in any of C_8^1 and C_8^2 having pairwise distances equal to 4 (for C_8^1: elements numbers 1, 3 and 5 from Table 4; and for C_8^2: elements 1, 2 and 9 from Table 5). By Dirichlet's principle at least two of these three elements belong to A_0 or A_1. The distance between the corresponding elements of A equals 4 and since $A_0 \cup A_1$ is equivalent to C_8^1 or C_8^2 we have a contradiction to Proposition 3.

Therefore a covering of Q_9 with 67 elements does not exist. Hence $T(9) \geq 67$.

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