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SOME NOTES ON ALMOST PARACONTACT STRUCTURES

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Abstract

In the present paper we study an almost paracontact Riemannian manifold which is closely related to the pure Riemannian metric. We construct a new tensor field corresponding to pure metric of almost paracontact structure. An integrability condition and curvature properties of structure by using this tensor field are presented. Finally, we study almost paracontact structures with structural exact 1-form.

Key words: almost paracontact structure, pure Riemannian metric, pure connection, Φ_φ-operator

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1. Introduction. Let M_n be a paracompact differentiable manifold of dimension n. We denote by $\Im_q^p(M_n)$ the set of all differentiable tensor fields of type (p, q) on M_n, and let $\varphi \in \Im_1^1(M_n)$, $\xi \in \Im_0^1(M_n)$ and $\eta \in \Im_1^0(M_n)$ be a tensor field of type $(1, 1)$, a vector field and 1-form on M_n respectively. If φ, ξ and η satisfy the conditions

$$\eta(\xi) = 1,$$

$$\varphi^2 X = X - \eta(X)\xi$$

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for any $X \in \Im_0^1(M_n)$, then M_n is said to have an almost paracontact structure (φ, ξ, η) and M_n is called an almost paracontact manifold [6, 7]. Then the equations

$$\begin{aligned} &\varphi\xi = 0, \\ (1) \qquad &\eta(\varphi X) = 0, \\ &\operatorname{rank} \varphi = n - 1 \end{aligned}$$

hold good. In [5] is introduced the notion of an almost paracontact Riemannian manifold of type (r, s), where r and s are the numbers of the multiplicity of the structural eigenvalues 1 and -1, respectively (for type of (n, n), see [1]).

Every almost paracontact manifold M_n admits an associated Riemannian metric tensor field g such that [6]

$$(2) \qquad \begin{aligned} g(X, \xi) &= \eta(X), \\ g(X, Y) - \eta(X)\eta(Y) &= g(\varphi X, \varphi Y) \end{aligned}$$

for any $X, Y \in \Im_0^1(M_n)$. The structure (φ, ξ, η, g) is called almost paracontact Riemannian structure on M_n.

Let M_n be an almost paracontact Riemannian manifold with structure (φ, ξ, η, g). We easily see that the Riemannian metric g is a pure tensor field with respect to φ, i.e.

$$(3) \qquad g(\varphi X, Y) = g(X, \varphi Y).$$

In fact, putting $Y = \varphi Z$, $Z \in \Im_0^1(M_n)$ in (2), by virtue of (1), it follows that g is pure

$$\begin{aligned} g(X, \varphi Z) &= g(X, \varphi Z) - \eta(X)\eta(\varphi Z) = g(\varphi X, \varphi^2 Z) \\ &= g(\varphi X, Z - \eta(Z)\xi) = g(\varphi X, Z) - g(\varphi X, \eta(Z)\xi) \\ &= g(\varphi X, Z) - \eta(Z)g(\varphi X, \xi) = g(\varphi X, Z) - \eta(Z)\eta(\varphi X) \\ &= g(\varphi X, Z). \end{aligned}$$

2. Almost Φ-paracontact Riemannian manifolds. On almost paracontact Riemannian manifold $(M_n, \varphi, \xi, \eta, g)$, we define an operator

$$\Phi_\varphi : \Im_2^0(M_n) \to \Im_3^0(M_n)$$

associated with φ and applied to a pure tensor field g by (see [4, 8])

$$(4) \qquad \begin{aligned} (\Phi_\varphi g)(X, Z_1, Z_2) = (\varphi X)(g(Z_1, Z_2)) - Xg(\varphi Z_1, Z_2) \\ + g((L_{Z_1}\varphi)X, Z_2) + g(Z_1, (L_{Z_2}\varphi)X), \end{aligned}$$

where L_Z is a Lie derivation with respect to the vector field Z.

If a pure tensor field g satisfies $g \in Ker\Phi_\varphi$ ($\Phi_\varphi g = 0$) for any tensor structure $\varphi \in \Im_1^1(M_n)$, then we shall call it a Φ-tensor field. If the structure tensor field φ is a complex structure, then a Φ-tensor field g is a holomorphic (an analytic) tensor field. If the structure tensor field φ is a product structure and g is a Φ-tensor field (i.e. $\Phi_\varphi g = 0$ which is equivalent to the equation $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g), then g is decomposable. There are several examples of Φ-tensor metrics (paraholomorphic B-metrics [2], hyperholomorphic B-metrics [3]).

Let now M_n be an almost paracontact Riemannian manifold with structure (φ, ξ, η, g). If the Riemannian metric g is a Φ-tensor field, then the almost paracontact Riemannian structure on M_n is called an almost Φ-paracontact structure and M_n is called an almost Φ-paracontact.

Now we give a characterization of almost Φ-paracontact Riemannian manifolds.

Theorem 1. *An almost paracontact Riemannian manifold M_n with structure (φ, ξ, η, g) is an almost Φ-paracontact if and only if φ is parallel with respect to the Levi-Civita connection ∇ of $g(\nabla\varphi = 0)$.*

Proof. Using (3) and $L_X Y = \nabla_X Y - \nabla_Y X$, from (4) we obtain

$$(5)\quad \begin{aligned}(\Phi_\varphi g)(X, Z_1, Z_2) &= (\varphi X)g(Z_1, Z_2) - Xg(\varphi Z_1, Z_2) + g(\nabla_{Z_1}\varphi X, Z_2)\\ &- g(\nabla_{\varphi X} Z_1, Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(\varphi(\nabla_X Z_1), Z_2)\\ &+ g(Z_1, \nabla_{Z_2}\varphi X) - g(Z_1, \nabla_{\varphi X} Z_2) - g(Z_1, \varphi(\nabla_{Z_2} X)) + g(Z_1, \varphi(\nabla_X Z_2)).\end{aligned}$$

On the other hand, we find

$$(6)\quad \begin{aligned}&g(\nabla_{Z_1}\varphi X, Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(Z_1, \nabla_{Z_2}\varphi X) - g(Z_1, \varphi(\nabla_{Z_2} X))\\ &\qquad = g((\nabla\varphi)(X, Z_1), Z_2) + g(Z_1, (\nabla\varphi)(X, Z_2)).\end{aligned}$$

Substituting (6) into (5), we have

$$(7)\quad \begin{aligned}(\Phi_\varphi g)(X, Z_1, Z_2) &= (\varphi X)g(Z_1, Z_2) - Xg(\varphi Z_1, Z_2) + g((\nabla\varphi)(X, Z_1), Z_2)\\ &+ g(Z_1, (\nabla\varphi)(X, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2)\\ &+ g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2).\end{aligned}$$

Since

$$(\varphi X)g(Z_1, Z_2) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2) = (\nabla_{\varphi X} g)(Z_1, Z_2) = 0,$$
$$-Xg(\varphi Z_1, Z_2) + g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2) = -g((\nabla_X\varphi)Z_1, Z_2),$$

equation (7) implies

$$(\Phi_\varphi g)(X, Z_1, Z_2) = -g((\nabla\varphi)(X, Z_1), Z_2) + g((\nabla\varphi)(Z_1, X), Z_2) + g(Z_1, (\nabla\varphi)(Z_2, X)). \tag{8}$$

Let $\nabla\varphi = 0$, then from (8) we have $\Phi_\varphi g = 0$.

Conversely, let now $\Phi_\varphi g = 0$. Similarly we obtain

$$(\Phi_\varphi g)(Z_2, Z_1, X) = -g((\nabla\varphi)(Z_2, Z_1), X) + g((\nabla\varphi)(Z_1, Z_2), X) + g(Z_1, (\nabla\varphi)(X, Z_2)). \tag{9}$$

Combining (8) and (9), by virtue of

$$g(Z, (\nabla\varphi)(Y, X)) = g((\nabla\varphi)(Y, Z), X)$$

we find

$$(\Phi_\varphi g)(X, Z_1, Z_2) + (\Phi_\varphi g)(Z_2, Z_1, X) = 2g(X, (\nabla_{Z_1}\varphi)Z_2). \tag{10}$$

Putting $\Phi_\varphi g = 0$ in (10), we find $\nabla\varphi = 0$. Thus Theorem 1 is proved. □

Let M_n be an almost paracontact Riemannian manifold with structure (φ, ξ, η, g). If the matrix of φ is constant on a certain set of natural frames $\{\partial_i\}$, then φ is said to be integrable. It is well-known that if there exists a torsion-free connection $\widetilde{\nabla}$ on M_n which satisfies the condition $\widetilde{\nabla}\varphi = 0$, then φ is integrable. Since the Levi-Civita connection ∇ is a torsion-free connection, from Theorem 1 we have:

Theorem 2. *If $(M_n, \varphi, \xi, \eta, g)$ is an almost Φ-paracontact, then φ is integrable.*

3. Curvature properties. Let $\varphi \in \Im_1^1(M)$ and $\widetilde{\nabla}$ be a torsion-free φ-connection, i.e. $\widetilde{\nabla}\varphi = 0$. We call this connection a pure connection with respect to φ. We denote the curvature tensor of the pure connection $\widetilde{\nabla}$ by R:

$$R(X, Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z, \quad X, Y, Z \in \Im_0^1(M).$$

Applying Ricci's identity to φ

$$\widetilde{\nabla}_X((\widetilde{\nabla}_Y\varphi)Z) - \widetilde{\nabla}_Y((\widetilde{\nabla}_Z\varphi)Z) = R(X, Y)\varphi Z - \varphi(R(X, Y)Z) + (\widetilde{\nabla}_{[X,Y]}\varphi)Z + (\widetilde{\nabla}_Y\varphi)(\widetilde{\nabla}_X\varphi) - (\widetilde{\nabla}_X\varphi)(\widetilde{\nabla}_Y\varphi),$$

we get

$$\varphi R(X, Y)Z = R(X, Y)\varphi Z$$

by virtue of $\widetilde{\nabla}\varphi = 0$. Thus, we have

Lemma 1. *Let $\varphi \in \Im_1^1(M)$. If $\widetilde{\nabla}$ is a pure connection with respect to φ, then*

$$\varphi R(X,Y)Z = R(X,Y)\varphi Z \tag{11}$$

for any $X, Y, Z \in \Im_0^1(M)$.

We denote by Φ_φ the operator defined by $(\Phi_\varphi X)Y = -(L_X\varphi)Y$ for any $X, Y \in \Im_0^1(M_n)$. Let now φ be invariant under the local transformations generated by X, Y, Z ($L_X\varphi = L_Y\varphi = L_Z\varphi = 0$), i.e. $X, Y, Z \in \operatorname{Ker}\Phi_\varphi$. We have

Lemma 2. *Let $\varphi \in \Im_1^1(M)$ and $\widetilde{\nabla}$ be a pure connection with respect to φ. The curvature tensor field R of $\widetilde{\nabla}$ is a pure tensor field with respect to φ ($\varphi R(X,Y)Z = R(\varphi X,Y)Z = R(X,\varphi Y)Z = R(X,Y)\varphi Z$) if and only if*

$$\Phi_{\varphi X}(\widetilde{\nabla}_Y Z) = \widetilde{\nabla}_{\varphi X}\widetilde{\nabla}_Y Z - \varphi(\widetilde{\nabla}_X\widetilde{\nabla}_Y Z) = 0$$

for any $X, Y, Z \in \operatorname{Ker}\Phi_\varphi$.

Proof. Since $X, Y, Z \in \operatorname{Ker}\Phi_\varphi$ and $\widetilde{\nabla}\varphi = 0$, $\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = 0$, we have

$$\begin{aligned} R(X,\varphi Y)Z &= \widetilde{\nabla}_X\widetilde{\nabla}_{\varphi Y}Z - \widetilde{\nabla}_{\varphi Y}\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,\varphi Y]}Z \\ &= \widetilde{\nabla}_X\varphi(\widetilde{\nabla}_Y Z) - \widetilde{\nabla}_{\varphi Y}\widetilde{\nabla}_X Z - \widetilde{\nabla}_{(L_X\varphi)Y+\varphi L_X Y}Z \\ &= \varphi(\widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z) + \varphi(\widetilde{\nabla}_Y\widetilde{\nabla}_X Z) - \widetilde{\nabla}_{\varphi Y}\widetilde{\nabla}_X Z \\ &= \varphi R(X,Y)Z - \Phi_{\varphi Y}(\widetilde{\nabla}_X Z). \end{aligned}$$

Thus

$$R(X,\varphi Y)Z = \varphi R(X,Y)Z - \Phi_{\varphi Y}\widetilde{\nabla}_X Z. \tag{12}$$

By virtue of $R(X,Y)Z = -R(Y,X)Z$, from (12) we have

$$R(\varphi X,Y)Z = -R(Y,\varphi X)Z = -\varphi R(Y,X)Z + \Phi_{\varphi X}(\widetilde{\nabla}_Y Z). \tag{13}$$

From (11)–(13) we see that the curvature tensor R of a pure connection $\widetilde{\nabla}$ is pure in all arguments if and only if

$$\Phi_{\varphi X}(\widetilde{\nabla}_Y Z) = 0$$

for any $X, Y, Z \in \operatorname{Ker}\Phi_\varphi$. Thus Lemma 2 is proved. □

Remark 1. From (13) we see that

$$\Phi_{\varphi X}(\widetilde{\nabla}_Y Z) \in \Im_3^1(M).$$

Therefore, in the sequel we write $(\Phi_\varphi\widetilde{\nabla})(X,Y,Z)$ for $\Phi_{\varphi X}(\widetilde{\nabla}_Y Z)$

$$(\Phi_\varphi\widetilde{\nabla})(X,Y,Z) = \widetilde{\nabla}_{\varphi X}\widetilde{\nabla}_Y Z - \varphi(\widetilde{\nabla}_X\widetilde{\nabla}_Y Z).$$

Thus $\Phi_\varphi\widetilde{\nabla}$ is a Φ_φ-operator applied to a pure connection $\widetilde{\nabla}$.

We will see that, if a pure torsion-free connection $\widetilde{\nabla}$ is a Levi-Civita connection of almost Φ-paracontact Riemannian manifolds, then immediately it follows that $\Phi_\varphi \widetilde{\nabla} = 0$.

Theorem 3. *Let M_n be an almost Φ-paracontact Riemannian manifold with structure (φ, ξ, η, g). Then $\nabla \in \operatorname{Ker} \Phi_\varphi$, where Φ_φ-operator applied to the pure Levi-Civita connection ∇ of g.*

Proof. Let $X, Z_1, Z_2 \in \operatorname{Ker} \Phi_\varphi$. For almost Φ-paracontact Riemannian manifolds $(\Phi_\varphi g = 0)$, from (4) we have

$$(\varphi X)(g(Z_1, Z_2)) = X g(\varphi Z_1, Z_2). \tag{14}$$

Since $X \in \operatorname{Ker} \Phi_\varphi$, by virtue of $\nabla \varphi = 0$, we find

$$\begin{aligned}(\Phi_\varphi X)Y &= -(L_X \varphi)Y = [\varphi Y, X] + \varphi[X, Y] \\ &= \nabla_{\varphi Y} X - \varphi \nabla_X Y + \varphi(\nabla_X Y - \nabla_Y X) \\ &= \nabla_{\varphi Y} X - \varphi(\nabla_Y X) = 0.\end{aligned} \tag{15}$$

If $X = X^i \partial_i$, $Y = \partial_k$, $Z_1 = \partial_{j_1}$, $Z_2 = \partial_{j_2}$, then (14) and (15) reduce to

$$X^i \varphi_i^m \partial_m g_{j_1 j_2} = X^m \partial_m (\varphi_{j_1}^s g_{s j_2}) \tag{16}$$

and

$$\varphi_k^s \partial_s X^i = \varphi_s^i \partial_k X^s, \tag{17}$$

respectively.

Differentiating (16), using (17), $\partial_k \varphi_j^i = 0$ (Theorem 2) and

$$\varphi_j^m \Gamma_{im}^k = \varphi_m^k \Gamma_{ij}^m = \varphi_i^m \Gamma_{mj}^k,$$

where

$$\Gamma_{ij}^k \partial_k = \nabla_{\partial_i} \partial_j,$$

we obtain

$$\varphi_i^m \partial_{lm}^2 g_{j_1 j_2} = \varphi_{j_1}^m \partial_{li}^2 g_{m j_2}. \tag{18}$$

On the other hand, from (18) and $2\Gamma_{ij}^k = g^{kt}(\partial_i g_{tj} + \partial_j g_{it} - \partial_t g_{ij})$, we have

$$\varphi_l^m \partial_m \Gamma_{ij}^k = \varphi_i^m \partial_l \Gamma_{mj}^k,$$

which is equivalent to the condition

$$(\Phi_\varphi \nabla)(X, Y, Z) = \nabla_{\varphi X} \nabla_Y W - \nabla_X \nabla_Y \varphi W = 0$$

with respect to the natural frame. □

From Lemma 2 and Theorem 3 we have

Theorem 4. *The curvature tensor field R of almost Φ-paracontact Riemannian manifold $(M_n, \varphi, \xi, \eta, g)$ is a pure tensor field with respect to φ.*

4. Almost paracontact manifolds with exact 1-form. Φ_φ-operator applied to an 1-form $\eta \in \Im_1^0(M)$ is defined by (see [4,8])

$$(\Phi_\varphi \eta)(X,Y) = (\varphi X)\eta(Y) - X\eta(\varphi Y) + \eta((L_Y\varphi)X) \tag{19}$$

for any $X, Y \in \Im_0^1(M)$.

Theorem 5. *Let M_n be an almost paracontact manifold with structure (φ, ξ, η, g). If η be an exact 1-form, then $\eta \in \operatorname{Ker}\Phi_\varphi$.*

Proof. Using

$$(d\eta)(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\}$$

for any $X, Y \in \Im_0^1(M)$, we have

$$\begin{aligned}(d\eta)(Y,\varphi X) &= \frac{1}{2}\{Y\eta(\varphi X) - (\varphi X)\eta(Y) - \eta([Y,\varphi X])\} \\ &= \frac{1}{2}\{Y\eta(\varphi X) - (\varphi X)\eta(Y) + \eta([\varphi X, Y] - \varphi[X,Y]) + \eta(\varphi[X,Y])\}.\end{aligned} \tag{20}$$

From (19) we have

$$(\Phi_\varphi \eta)(X,Y) = (\varphi X)\eta(Y) - X\eta(\varphi Y) - \eta([\varphi X, Y] - \varphi[X,Y]). \tag{21}$$

Substituting (21) into (20) we obtain

$$\begin{aligned}(d\eta)(Y,\varphi X) &= \frac{1}{2}\{-(\Phi_\varphi\eta)(X,Y) + Y\eta(\varphi X) - X\eta(\varphi Y) + \eta(\varphi[X,Y])\} \\ &= \frac{1}{2}\{-(\Phi_\varphi\eta)(X,Y) + Y(\eta\circ\varphi)(X) - X(\eta\circ\varphi)(Y) - (\eta\circ\varphi)([Y,X])\} \\ &= -\frac{1}{2}(\Phi_\varphi\eta)(X,Y) + (d(\eta\circ\varphi))(Y,X),\end{aligned}$$

where $(\eta\circ\varphi)(X) = \eta(\varphi X)$. From here, by virtue of $(\eta\circ\varphi)(X) = \eta(\varphi X) = 0$ (see (1)), we see that the last equation is equivalent to

$$(\Phi_\varphi\eta)(X,Y) = -2(d\eta)(Y,\varphi X),$$

which for $\eta = df$, $f \in \Im_0^0(M)$ turns into the following form:

$$(\Phi_\varphi\eta)(X,Y) = (\Phi_\varphi df)(X,Y) = (d^2 f)(Y,\varphi X) = 0. \qquad \square$$

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