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TWISTOR SPACES WITH POSITIVE HOLOMORPHIC BISECTIONAL CURVATURE

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Abstract

In this paper we provide twistorial examples of compact Hermitian manifolds with positive holomorphic bisectional curvature. We also observe that the so-called "squashed" metric on $\mathbb{CP}^3$, the twistor space of the sphere S^4, is a non-Kähler Hermitian–Einstein metric of positive holomorphic bisectional curvature, thus showing that a recent result of Kalafat and Koca in complex dimension two cannot be extended in higher dimensions.
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By a result of GOLDBERG and KOBAYASHI [6] the only n-dimensional compact connected Kähler manifold with an Einstein (or constant scalar curvature) metric of positive holomorphic bisectional curvature is the complex projective space $\mathbb{CP}^n$ with the Fubini–Study metric (up to rescaling). For complex dimension two, KALAFAT and KOCA [9] have recently relaxed the Kähler condition to the Hermitian one. The main purpose of the present note is to show that this is no longer true in complex dimensions greater than two.

Let M be a (connected) oriented Riemannian 4-manifold with metric g. Its curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of Λ^2TM defined by

$$g(\mathcal{R}(X\wedge Y), Z\wedge T) = g(R(X,Y)Z,T), \quad X,Y,Z,T\in TM,$$

where we adopt the following definition for the Riemannian curvature tensor R of g: $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

The Hodge star operator defines an endomorphism $*$ of $\Lambda^2 TM$ with $*^2 = \mathrm{Id}$. This endomorphism yields the decomposition

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM,$$

where $\Lambda^2_\pm TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1)-eigenvectors of $*$. The block-matrix of $\mathcal{R}$ with respect to this splitting of $\Lambda^2 TM$ is

$$\mathcal{R} = \begin{bmatrix} \frac{s}{6}\,\mathrm{Id} + \mathcal{W}_+ & \mathcal{B} \\ {}^t\mathcal{B} & \frac{s}{6}\,\mathrm{Id} + \mathcal{W}_- \end{bmatrix},$$

where s is the scalar curvature of g, $\mathcal{B}$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ represent the traceless Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The twistor space of M is the subbundle $\mathcal{Z}$ of $\Lambda^2_- TM$ consisting of all unit vectors. The Riemannian connection ∇ of M gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. More precisely, let $\pi : \Lambda^2_- TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in \mathcal{Z}$ is $\mathcal{V}_\sigma = \mathrm{Ker}(\pi|\mathcal{Z})_{*\sigma}$ ($T_\sigma\mathcal{Z}$ is always considered as a subspace of $T_\sigma(\Lambda^2_- TM)$). Note that $\mathcal{V}_\sigma$ consists of those vectors of $T_\sigma\mathcal{Z}$ that are tangent to the fibre $\mathcal{Z}_p = \pi^{-1}(p) \cap \mathcal{Z}$, $p = \pi(\sigma)$, of $\mathcal{Z}$ through the point σ. Since $\mathcal{Z}_p$ is the unit sphere in the vector space $\Lambda^2_- T_pM$, $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\Lambda^2_- T_pM$. Let s be a local section of $\mathcal{Z}$ such that $s(p) = \sigma$. Since s has a constant length, $\nabla_X s \in \mathcal{V}_\sigma$ for all $X \in T_pM$. Given $X \in T_pM$, the vector $X^h_\sigma = s_* X - \nabla_X s \in T_\sigma\mathcal{Z}$ depends only on p and σ. By definition, the horizontal space at σ is $\mathcal{H}_\sigma = \{X^h_\sigma : X \in T_pM\}$. Note that the map $X \to X^h_\sigma$ is an isomorphism between T_pM and $\mathcal{H}_\sigma$ with inverse map $\pi_* \mid \mathcal{H}_\sigma$.

Each point $\sigma \in \mathcal{Z}$ defines a complex structure K_σ on T_pM by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y), \quad X, Y \in T_pM. \tag{1}$$

Note that K_σ is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of K_σ.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_\pm T_pM$, $p \in M$. Following [1] and [5], define two almost-complex structures J_1 and J_2 on $\mathcal{Z}$ by

$$J_n V = (-1)^n \sigma \times V \quad \text{for} \quad V \in \mathcal{V}_\sigma$$
$$J_n X^h_\sigma = (K_\sigma X)^h_\sigma \quad \text{for} \quad X \in T_pM, \quad p = \pi(\sigma).$$

It is well-known [1] that J_1 is integrable (i.e. comes from a complex structure) if and only if M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable [5].

Let h_t be the Riemannian metric on $\mathcal{Z}$ given by

$$h_t = \pi^* g + t g^v,$$

where $t > 0$, g is the metric of M and g^v is the restriction of the metric of $\Lambda^2 TM$ on the vertical distribution $\mathcal{V}$. Then $\pi : (Z, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres and the almost-complex structures J_1 and J_2 are compatible with the metrics h_t.

Recall that the holomorphic bisectional curvature of an almost Hermitian manifold (N, h, J) is defined by

$$H(X, Y) = R(X, JX, Y, JY), \quad X, Y \in TM, \quad \|X\| = \|Y\| = 1,$$

where R is the Riemannian curvature tensor of the metric h [6]. Thus the holomophic sectional curvature of N is $H(X) = H(X, X)$.

Denote by $H_{t,n}(E, F)$ the holomorphic bisectional curvature of the twistor space $(\mathcal{Z}, h_t, J_n)$, $n = 1, 2$.

Lemma 1. *Let (M, g) be an oriented Riemannian 4-manifold of constant sectional curvature and scalar curvature s, and let $E, F \in T_\sigma \mathcal{Z}$ be arbitrary h_t-unit tangent vectors. Set $X = \pi_* E$, $Y = \pi_* F$, $V = \mathcal{V}E$, $W = \mathcal{V}F$. Then*

$$\begin{aligned}
H_{t,n}(E, F) = {} & \frac{s}{12}(g(X, Y)^2 + g(K_\sigma X, Y)^2) + t\|V\|^2\|W\|^2 \\
& + 2t\left(\frac{s}{24}\right)^2 (\|X\|^2\|Y\|^2 - g(K_\sigma X, Y)^2 - g(X, Y)^2) \\
& + (-1)^n \left(2\left(\frac{ts}{24}\right)^2 - \frac{ts}{12}\right)(\|X\|^2\|W\|^2 + \|Y\|^2\|V\|^2) \\
& + \left(2\left(\frac{ts}{24}\right)^2 (1 + (-1)^n) - \frac{ts}{12}\right)(g(K_\sigma X, Y) g(\sigma \times V, W) \\
& + (-1)^n g(X, Y) g(V, W)),
\end{aligned} \tag{2}$$

where K_σ is the complex structure on $T_{\pi(\sigma)}M$ determined by σ via (1).

Proof. An explicit formula for the sectional curvature of $(\mathcal{Z}, h_t)$ in terms of the curvature of the base manifold (M, g) has been obtained in [4]. Since M is of constant curvature, we have

$$g(R(X, K_\sigma X)Y, K_\sigma Y) = \frac{s}{6} g(X \wedge K_\sigma X, Y \wedge K_\sigma Y) = \frac{s}{12}(g(X, Y)^2 + g(K_\sigma X, Y)^2).$$

Then one can easily obtain the desired formula for the holomorphic bisectional curvature $H_{t,n}(E, F)$. □

Proposition 1. *The holomorphic bisectional curvature of the twistor space $(\mathcal{Z}, h_t, J_n)$ of an oriented Riemannian 4-manifold (M, g) is never constant.*

Proof. Suppose on the contrary that $(\mathcal{Z}, h_t, J_1)$ has constant holomorphic bisectional curvature κ. Then $(\mathcal{Z}, h_t, J_1)$ has constant holomorphic sectional curvature κ. According to ([4], Proposition 5.2), the metric g is of constant sectional curvature $\kappa = 1/t$. Thus, since $s = 12\kappa$, we have $st = 12$. Setting $st = 12$ and $n = 1$ in (2), we get

$$\begin{aligned}
H_{t,1}(E,F) &= \frac{1}{t}[g(X,Y)^2 + g(K_\sigma X,Y)^2] + t\|V\|^2\|W\|^2 \\
&+ \frac{1}{2t}[\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(K_\sigma X,Y)^2] + \frac{1}{2}[\|X\|^2\|W\|^2 + \|Y\|^2\|V\|^2] \\
&- g(K_\sigma X,Y)g(\sigma \times V,W) + g(X,Y)g(V,W) \\
&= \frac{1}{2t}[(g(X,Y) + tg(V,W))^2 + (g(X,K_\sigma Y) - tg(V,\sigma \times W))^2 \\
&+ (g(X,X) + tg(V,V))(g(Y,Y) + tg(W,W))] \\
&= \frac{1}{2t}[h_t(E,F)^2 + h_t(E,J_1F)^2 + 1].
\end{aligned}$$

This is not constant because $h_t(E,F)^2 + h_t(E,J_1F)^2$ is not constant, which contradicts our assumption.

According to ([4], Proposition 5.2), the holomorphic sectional curvature of $(\mathcal{Z}, h_t, J_2)$ is never constant, therefore the holomorphic bisectional curvature of $(\mathcal{Z}, h_t, J_2)$ is never constant, too. □

Proposition 2. *Let (M,g) be a Riemannian 4-manifold of constant sectional curvature.*

(i) *The holomorphic bisectional curvature of $(\mathcal{Z}, h_t, J_1)$ is positive if and only if $0 < ts < 24$.*

(ii) *If (M,g) is a flat manifold, the holomorphic bisectional curvature of $(\mathcal{Z}, h_t, J_n)$ is non-negative for every $t > 0$.*

Proof. To prove (i) we first note that

$$\|X\|^2.\|Y\|^2 - g(X,Y)^2 - g(K_\sigma X,Y)^2 = \|X\|^2.\|Z\|^2 \geq 0,$$

where Z is the orthogonal projection of Y on the orthogonal complement of the vector space $\mathrm{Span}(X, K_\sigma X)$. Now suppose that $0 < ts < 24$. Then using the Cauchy–Schwarz inequality we get

$$\frac{1}{2}\left(\frac{s}{6}g(X,Y)^2 + t\|V\|^2.\|W\|^2\right) \geq \sqrt{\frac{ts}{6}}|g(X,Y)||g(V,W)|$$

and

$$\frac{1}{2}\left(\frac{s}{6}g(K_\sigma X,Y)^2 + t\|V\|^2.\|W\|^2\right) \geq \sqrt{\frac{ts}{6}}|g(K_\sigma X,Y)||g(\sigma \times V,W)|.$$

We also have

$$\frac{1}{2}(\|X\|^2.\|W\|^2 + \|Y\|^2.\|V\|^2) \geq |g(X,Y)||g(V,W)|$$

and

$$\frac{1}{2}(\|X\|^2.\|W\|^2 + \|Y\|^2.\|V\|^2) \geq |g(K_\sigma X,Y)||g(\sigma \times V,W)|.$$

Since $0 < ts < 24$, it follows from (2) and the inequalities above that

$$H_{t,1}(E,F)$$

$$\geq \left(\sqrt{\frac{ts}{6}} - \frac{1}{2}\left(\frac{ts}{12}\right)^2\right)(|g(K_\sigma X,Y)||g(\sigma \times V,W)| + |g(X,Y)||g(V,W)|) \geq 0.$$

It is easy to see that the inequality $H_{t,1}(E,F) \geq 0$ is strict since otherwise either $E = 0$ or $F = 0$.

Finally, using (2) we get

$$H_{t,1}(X^h,W) = \left(\frac{ts}{12} - \frac{1}{2}\left(\frac{ts}{12}\right)^2\right)\|X\|^2\|W\|^2 \leq 0$$

for $ts \leq 0$ or $ts \geq 24$. This proves (i).

Part (ii) follows from (2) since, if the metric g is flat, we have

$$H_{t,n}(E,F) = t\|V\|^2.\|W\|^2 \geq 0. \qquad \square$$

Now note that the metric h_t on the twistor space Z of a self-dual Einstein manifold M with positive scalar curvature s for $t = \dfrac{6}{s}$ is Einsten [7] and is not Kähler with respect to the complex structure J_1 [8]. If we take $M = S^4$ with the round metric, then $s = 12$, $(Z, J_1) = \mathbb{CP}^3$ and $h_{1/2}$ is the so-called "squashed" metric ([2], Example 9.83). Thus, by Proposition 2, $\mathbb{CP}^3$ with the "squashed" metric and the standard complex structure is an Einstein Hermitian manifold of positive holomorphic bisectional curvature which is not isometric to $\mathbb{CP}^3$ with a rescaling of the Fubini–Study metric. This shows that the result of Kalafat and Koca mentioned above cannot be extended in complex dimension three.

It is well-known [1] that the twistor space (Z, J_1) of a torus T^4 is biholomorphic to $T^4 \times S^2$ with the complex structure defined in ([3], Example de fibration au sense de la proposition I.2.2). By Proposition 2, $T^4 \times S^2$ admits a one-parameter family of Hermitian metrics of non-negative holomorphic bisectional curvature.

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