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DETERMINANT AND NORM INEQUALITIES FOR BIALTERNATE PRODUCTS OF MATRICES

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Abstract

The paper presents further results on the properties of two special matrix products recently studied in [1]. In particular, we derive inequalities involving the determinant and matrix norms of the bialternate and permanental bialternate products of square matrices. Special attention is paid to the case of Hermitian matrices where both products preserve the Hermitian property of the multiplied factors.

Key words: inequalities, bialternate product, Kronecker product

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1. Introduction. In [1], the well-known bialternate product of two square matrices is re-examined and another matrix product with similar properties is also defined and studied. Basic characteristics of both products are established in this reference including relations with compound and induced matrices, spectral and mixed-product properties, trace equalities. An important connection with the Kronecker product of matrices is also given. It should be noted that specific spectral properties of matrices constructed by means of these matrix products provide an efficient tool in studying stability problems and in detecting and computing Hopf bifurcations in systems of ordinary differential equations, e.g. see [1–3] and references therein.

In the present work, we establish relations involving the two bialternate products in the special case where matrix factors are Hermitian matrices. A basic observation in this case shows that both products preserve the Hermitian property of the multiplied matrices. Motivated by this fact, we provide inequalities involving the determinant and the Shatten p-norms of the matrix products. The obtained results are essentially based on the work in [1] and show that well-known properties of the Kronecker product can be used in order to derive certain properties of the bialternate and the permanental bialternate products of matrices.

2. Notation and preliminaries. The set of $m \times n$ matrices with complex elements will be denoted by $M_{m,n}(\mathbb{C})$ or $M_n(\mathbb{C})$ if $m = n$. The identity matrix with an appropriate size will be denoted by I.

We consider matrix products defined as follows.

Definition 1 ([1,4]). The bialternate product of $A = [a_{ij}] \in M_n(\mathbb{C})$ and $B = [b_{ij}] \in M_n(\mathbb{C})$ for $n > 1$ is defined to be the matrix $F = A \cdot B$ where the entries of F are given by

$$(1) \quad f_{ij,kl} = \frac{1}{2}\left(\det\begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{bmatrix} + \det\begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{bmatrix}\right), \quad 1 \le i < j \le n,\ 1 \le k < l \le n.$$

In the next definition, the permanent of a matrix is denoted by 'per' and a quantity $\mu(p,q)$ is used with values determined for positive integers p and q as $\mu(p,q) = 2$ if $p = q$ and $\mu(p,q) = 1$ if $p \ne q$. Thus, $\mu(p,q) = 1 + \delta_{p,q}$, where $\delta_{p,q}$ is the Kronecker symbol.

Definition 2 ([1]). The permanental bialternate product of $A = [a_{ij}] \in M_n(\mathbb{C})$ and $B = [b_{ij}] \in M_n(\mathbb{C})$ is defined to be the matrix $G = A \times B$, where the entries of G are given by

$$(2) \quad g_{ij,kl} = \frac{1}{2\sqrt{\mu(i,j)\mu(k,l)}}\left(\operatorname{per}\begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{bmatrix} + \operatorname{per}\begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{bmatrix}\right), \quad \begin{matrix} 1 \le i \le j \le n, \\ 1 \le k \le l \le n. \end{matrix}$$

In the above definitions, we have used double indices ij to label the rows and double indices kl to label the columns of the product matrices. Thus, $F \in M_{\binom{n}{2}}(\mathbb{C})$ and $G \in M_{\binom{n+1}{2}}(\mathbb{C})$.

In order to illustrate the two products, let $n = 2$, $A = [a_{ij}] \in M_2(\mathbb{C})$ and $B = [b_{ij}] \in M_2(\mathbb{C})$. In this case the bialternate product $F = A \cdot B$ is a scalar given by

$$F = A \cdot B = f_{12,12} = \frac{1}{2}(a_{11}b_{22} + b_{11}a_{22} - a_{12}b_{21} - b_{12}a_{21}).$$

The permanental bialternate product is a 3×3 matrix, i.e.

$$G = A \times B = \begin{bmatrix} g_{11,11} & g_{11,12} & g_{11,22} \\ g_{12,11} & g_{12,12} & g_{12,22} \\ g_{22,11} & g_{22,12} & g_{22,22} \end{bmatrix},$$

where

$$
\begin{aligned}
g_{11,11} &= a_{11}b_{11};\\
g_{11,12} &= \frac{1}{\sqrt{2}}(a_{11}b_{12} + b_{11}a_{12});\\
g_{11,22} &= a_{12}b_{12};\\
g_{12,11} &= \frac{1}{\sqrt{2}}(a_{11}b_{21} + b_{11}a_{21});\\
g_{12,12} &= \frac{1}{2}(a_{11}b_{22} + a_{12}b_{21} + b_{11}a_{22} + b_{12}a_{21});\\
g_{12,22} &= \frac{1}{\sqrt{2}}(a_{12}b_{22} + b_{12}a_{22});\\
g_{22,11} &= a_{21}b_{21};\\
g_{22,12} &= \frac{1}{\sqrt{2}}(a_{21}b_{22} + b_{21}a_{22});\\
g_{22,22} &= a_{22}b_{22}.
\end{aligned}
$$

Obviously, both operations '$\cdot$' and '$\times$' are bilinear and commutative. According to (1) and (2), if $A = B$ then

$$A \cdot A = C_2(A) \text{ and } A \times A = P_2(A), \tag{3}$$

where $C_2(A)$ and $P_2(A)$ denote the second compound and the second induced matrices of A, respectively, e.g. see [5]. Furthermore, products $2(A \cdot B)$ and $2(A \times B)$ can be expressed in terms of the second compound and second induced matrices as follows, [1]:

$$2(A \cdot B) = C_2(A + B) - C_2(A) - C_2(B), \tag{4}$$

$$2(A \times B) = P_2(A + B) - P_2(A) - P_2(B). \tag{5}$$

Another simple and useful property is easily obtained from (1) and (2). In particular, if A and B are upper triangular (lower triangular, diagonal) matrices with diagonal entries a_{ii} and b_{ii}, $i = 1, \ldots, n$, respectively, then $F = A \cdot B$ and $G = A \times B$ are upper triangular (lower triangular, diagonal) matrices with diagonal entries

$$f_{ij,ij} = \frac{1}{2}(a_{ii}b_{jj} + b_{ii}a_{jj}), \quad 1 \le i < j \le n \tag{6}$$

and

$$g_{ij,ij} = \frac{1}{2}(a_{ii}b_{jj} + b_{ii}a_{jj}), \quad 1 \le i \le j \le n, \tag{7}$$

respectively.

In the sequel, we shall also use the well-known Kronecker product of matrices. It is denoted by $\otimes$ and if $X \in M_{m,n}(\mathbb{C})$, $Y \in M_{p,q}(\mathbb{C})$ then $X \otimes Y \in M_{mp,nq}(\mathbb{C})$, e.g. see [6]. It should be noted that no special symbol will be used to identify the standard product, i.e. if A and B are matrices with appropriate sizes, their product in the usual sense will be denoted by AB.

The next results [1] represent important properties of the bialternate and the permanental bialternate products.

Theorem 1. *The following equalities hold for matrices* $A, B, C, D \in M_n(\mathbb{C})$:

$$(A \cdot B)(C \cdot D) = \frac{1}{2}(AC \cdot BD + AD \cdot BC), \tag{8}$$

$$(A \times B)(C \times D) = \frac{1}{2}(AC \times BD + AD \times BC). \tag{9}$$

Theorem 2. *Let* $A, B \in M_n(\mathbb{C})$. *There is an orthogonal matrix* $U \in M_{n^2}(\mathbb{R})$ *such that*

$$U^T(A \otimes B + B \otimes A)U = 2\left[\begin{array}{c|c} A \cdot B & 0 \\ \hline 0 & A \times B \end{array}\right]. \tag{10}$$

The mixed-product property given by (8) and (9) in Theorem 1 relates each of the products (1) and (2) with the standard matrix product. Theorem 2 provides a simple and elegant relation with the Kronecker product of matrices.

3. Results. We begin with a basic observation concerning bialternate and permanental bialternate products of Hermitian matrices.

Proposition 1. *If* $A = [a_{ij}] \in M_n(\mathbb{C})$ *and* $B = [b_{ij}] \in M_n(\mathbb{C})$ *are Hermitian (positive semidefinite, positive definite) matrices then* $F = A \cdot B$ *and* $G = A \times B$ *are also Hermitian (positive semidefinite, positive definite) matrices.*

Proof. Since $a_{ij} = \bar{a}_{ji}$ and $b_{ij} = \bar{b}_{ji}$, $1 \le i, j \le n$, it is easily seen from (1) and (2) that $f_{ij,kl} = \bar{f}_{kl,ij}$ for $1 \le i < j \le n$, $1 \le k < l \le n$ and $g_{ij,kl} = \bar{g}_{kl,ij}$ for $1 \le i \le j \le n$, $1 \le k \le l \le n$ which shows that both F and G are Hermitian matrices. If, in addition, A and B are positive semidefinite (positive definite), then their Kronecker product is a positive semidefinite (positive definite) matrix [6]. In this case, the left-hand side of (10) is a positive semidefinite (positive definite) matrix which implies that both $A \cdot B$ and $A \times B$ are positive semidefinite (positive definite) matrices. $\square$

In what follows, we present results involving the determinants of $A \cdot B$ and $A \times B$.

Lemma 1. *Let* $A, B \in M_n(\mathbb{C})$ *with* $\det A \neq 0$ *and* $\det B \neq 0$. *Then*

$$(\det(A \cdot B))^2 = \left(\frac{1}{2}\right)^{\binom{n}{2}} (\det A)^{n-1} (\det B)^{n-1} \prod_{1 \le i < j \le n} \frac{1}{2} \frac{(\lambda_i + \lambda_j)^2}{\lambda_i \lambda_j}, \tag{11}$$

$$(\det(A \times B))^2 = \left(\frac{1}{2}\right)^{\binom{n+1}{2}} (\det A)^{n+1} (\det B)^{n+1} \prod_{1 \le i \le j \le n} \frac{1}{2} \frac{(\lambda_i + \lambda_j)^2}{\lambda_i \lambda_j}, \tag{12}$$

where λ_i, $1 \le i \le n$, *are the eigenvalues of* AB^{-1}.

Proof. By using the mixed product property (8), we have

$$A \cdot B = (A \cdot A)(I \cdot A^{-1}B) \text{ and } A \cdot B = (B \cdot B)(B^{-1}A \cdot I) \tag{13}$$

and hence,

$$\begin{aligned}(\det(A \cdot B))^2 &= \det(A \cdot A)\det(B \cdot B)\det\left(\frac{1}{2}(B^{-1}A \cdot A^{-1}B + I)\right)\\ &= \left(\frac{1}{2}\right)^{\binom{n}{2}} (\det A)^{n-1}(\det B)^{n-1}\det(B^{-1}A \cdot A^{-1}B + I).\end{aligned} \tag{14}$$

In (14), we have used (3) and the fact that $\det C_2(A) = (\det A)^{n-1}$, [5]. By Schur's triangulation theorem, there is a unitary matrix $U \in M_n(\mathbb{C})$ such that $T = U^* A^{-1} B U$ is an upper triangular matrix. Again using (8), it is obtained

$$(U^* \cdot U)(B^{-1}A \cdot A^{-1}B + I)(U \cdot U) = T^{-1} \cdot T + I. \tag{15}$$

In view of (6), the matrix in the right-hand side of (15) is an upper triangular matrix with diagonal entries given by

$$\frac{1}{2}(\lambda_i^{-1}\lambda_j + \lambda_i\lambda_j^{-1}) + 1 = \frac{1}{2}\frac{(\lambda_i + \lambda_j)^2}{\lambda_i\lambda_j}. \tag{16}$$

Thus, (11) follows from (14), (15) and (16). Equality (12) is obtained in a similar way by using the mixed product property (9). □

Theorem 3. *Let* $A = [a_{ij}] \in M_n(\mathbb{C})$ *and* $B = [b_{ij}] \in M_n(\mathbb{C})$ *be Hermitian positive semidefinite matrices. The following inequalities hold:*

$$(\det(A \cdot B))^2 \ge (\det A)^{n-1}(\det B)^{n-1}; \tag{17}$$

$$(\det(A \times B))^2 \ge (\det A)^{n+1}(\det B)^{n+1}; \tag{18}$$

$$\det(A \cdot B) \le \left(\frac{1}{2}\right)^{\binom{n}{2}} \prod_{1 \le i < j \le n} \left((a_{ii}b_{jj})^{\frac{1}{2}} + (b_{ii}a_{jj})^{\frac{1}{2}}\right)^2; \tag{19}$$

$$\det(A \times B) \le \left(\frac{1}{2}\right)^{\binom{n+1}{2}} \prod_{1 \le i \le j \le n} \left((a_{ii}b_{jj})^{\frac{1}{2}} + (b_{ii}a_{jj})^{\frac{1}{2}}\right)^2. \tag{20}$$

Proof. By Proposition 1, $A \cdot B$ and $A \times B$ are positive semidefinite and if either $\det A = 0$ or $\det B = 0$, then inequalities (17) and (18) are trivial. Let $\det A > 0$, $\det B > 0$ and denote by λ_i, $1 \le i \le n$ the eigenvalues of AB^{-1}. Since both A and B^{-1} are positive definite, it is well-known [7] that $\lambda_i > 0$, $1 \le i \le n$. Now, by using the inequality $(\lambda_i\lambda_j)^{\frac{1}{2}} \le \frac{1}{2}(\lambda_i + \lambda_j)$, $1 \le i, j \le n$, in (11) and (12), we obtain (17) and (18), respectively.

Inequality (19) is obvious if $\det(A\cdot B)=0$ and also, (20) is obvious if $\det(A\times B)=0$. Assume that $\det(A\cdot B)>0$, $\det(A\times B)>0$ and let $f_{ij,ij}$ denote the diagonal entries of $A\cdot B$ for $1\le i<j\le n$ and $g_{ij,ij}$ denote the diagonal entries of $A\times B$ for $1\le i\le j\le n$. Then, $f_{ij,ij}>0$, $g_{ij,ij}>0$ and from (1), we have

$$\begin{aligned}f_{ij,ij}&=\frac{1}{2}(a_{ii}b_{jj}-a_{ij}b_{ji}+b_{ii}a_{jj}-b_{ij}a_{ji})\le\frac{1}{2}(a_{ii}b_{jj}+b_{ii}a_{jj}+2|a_{ij}|\,|b_{ij}|)\\&\le\frac{1}{2}\left(a_{ii}b_{jj}+b_{ii}a_{jj}+2(a_{ii}a_{jj}b_{ii}b_{jj})^{\frac{1}{2}}\right)=\frac{1}{2}\left((a_{ii}b_{jj})^{\frac{1}{2}}+(b_{ii}a_{jj})^{\frac{1}{2}}\right)^2.\end{aligned}\tag{21}$$

In (21), we have used the fact that $a_{ij}=\bar{a}_{ji}$, $b_{ij}=\bar{b}_{ji}$ and inequalities $a_{ii}a_{jj}-|a_{ij}|^2\ge 0$ and $b_{ii}b_{jj}-|b_{ij}|^2\ge 0$, $1\le i<j\le n$ following from the positive semidefiniteness of the principal submatrices of A and B. Similarly, the diagonal entries of $A\times B$ satisfy

$$g_{ij,ij}=\frac{1}{2}(a_{ii}b_{jj}+a_{ij}b_{ji}+b_{ii}a_{jj}+b_{ij}a_{ji})\le\frac{1}{2}\left((a_{ii}b_{jj})^{\frac{1}{2}}+(b_{ii}a_{jj})^{\frac{1}{2}}\right)^2.\tag{22}$$

By the Hadamard's determinantal inequality [7], we have

$$\det(A\cdot B)\le\prod_{1\le i<j\le n}f_{ij,ij};\tag{23}$$

$$\det(A\times B)\le\prod_{1\le i\le j\le n}g_{ij,ij},\tag{24}$$

which, together with (21) and (22), imply (19) and (20). □

Corollary 1. *Let $A=[a_{ij}]\in M_n(\mathbb{C})$ and $B=[b_{ij}]\in M_n(\mathbb{C})$ be Hermitian positive semidefinite matrices. The following inequalities hold*:

$$\det(A\cdot B)\det(A\times B)\ge(\det(AB))^n\tag{25}$$

$$2^{n^2}\det(A\cdot B)\det(A\times B)\le\prod_{i=1}^{n}\prod_{j=1}^{n}(a_{ii}b_{jj}+b_{ii}a_{jj}).\tag{26}$$

Proof. Inequality (25) follows from Theorem 3 directly. Also, by (10), we have

$$\det(A\otimes B+B\otimes A)=2^{n^2}\det(A\cdot B)\det(A\times B).\tag{27}$$

In order to show (26), we note that the diagonal entries of $A\otimes B+B\otimes A$ are $a_{ii}b_{jj}+b_{ii}a_{jj}$, $1\le i,j\le n$ and by applying the Hadamard's inequality in (27), we obtain (26). □

In relation with the above corollary, we note that (19) and (20) directly imply that

$$\det(A \cdot B)\det(A \times B) \leq 2^{-n^2} \prod_{i=1}^{n} \prod_{j=1}^{n} \left((a_{ii}b_{jj})^{\frac{1}{2}} + (b_{ii}a_{jj})^{\frac{1}{2}} \right)^2 . \tag{28}$$

However, it is obvious that (26) is stronger than (28).

Next, we give bounds for norms and spectral radii of $A \cdot B$ and $A \times B$. The spectral radius of a matrix $A \in M_n(\mathbb{C})$ will be denoted by $\rho(A)$. By $\|A\|_p$ we shall denote the unitarily invariant Shatten p-norm of A defined as the Hölder p-norm of the vector of singular values of A for $p \in [1, \infty)$, e.g. see [7]. We recall that for $p = 1, 2$ $\|A\|_p$ is the general trace norm and Frobenius (Euclidean) norm of A, respectively. Also, for $p \to \infty$, we obtain the spectral norm of A.

Theorem 4. *Let $A, B \in M_n(\mathbb{C})$. Then*

$$\max\{\|A \cdot B\|_p, \|A \times B\|_p\} \leq \|A\|_p \|B\|_p. \tag{29}$$

If A and B are Hermitian, then

$$\max\{\rho(A \cdot B), \rho(A \times B)\} \leq \rho(A)\rho(B). \tag{30}$$

Proof. It is easily seen that

$$\max\{\|A \cdot B\|_p, \|A \times B\|_p\} \leq \left\| \left[\begin{array}{c|c} A \cdot B & 0 \\ \hline 0 & A \times B \end{array} \right] \right\|_p \tag{31}$$

and by Theorem 2, it follows that

$$\max\{\|A \cdot B\|_p, \|A \times B\|_p\} \leq \frac{1}{2}(\|A \otimes B\|_p + \|B \otimes A\|_p). \tag{32}$$

Using the well-known relations $\|A \otimes B\|_p = \|B \otimes A\|_p = \|A\|_p \|B\|_p$ in (32), we obtain 29. By Proposition 1, if A and B are Hermitian matrices then $A \cdot B$ and $A \times B$ are also Hermitian and in this case, inequality 30 follows from 29 with $p \to \infty$ and the fact that the spectral radius and the spectral norm of a Hermitian matrix are equal. □

Theorem 4 establishes the sub-multiplicativity property of the Shatten p-norms with respect to the two bilaternate products, i.e.

$$\|A \cdot B\|_p \leq \|A\|_p \|B\|_p \text{ and } \|A \times B\|_p \leq \|A\|_p \|B\|_p. \tag{33}$$

By using (33), we can also obtain inequalities

$$\|A \cdot I + B \cdot I\|_p \leq \|A\|_p \|I\|_p + \|B\|_p \|I\|_p,$$
$$\|A \times I + B \times I\|_p \leq \|A\|_p \|I\|_p + \|B\|_p \|I\|_p,$$

which can be viewed as analogues of the triangle inequality.

4. Conclusion. In Theorems 3 and 4, we have obtained determinant and norm inequalities which present some new properties of the bialternate and the permanental bilaternate products of matrices. The proofs are based on the results from [1] and especially on the mixed-product property (Theorem 1) and the relation with the Kronecker product of matrices (Theorem 2). The obtained results suggest that the basic properties of the two bialternate products, given by equalities (8), (9) and (10), can be further employed in order to obtain counterparts of well-known inequalities involving Kronecker and Hadamard products of matrices. Chapters 4 and 5 in reference [6] provide a wide variety of such inequalities.

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