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GRADIENT ESTIMATES AND GRADIENT BLOW UP OF VISCOSITY SOLUTIONS TO DEGENERATE ELLIPTIC EQUATIONS ON "INTERIOR BOUNDARIES"

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Abstract

Sharp conditions for gradient estimates or gradient blow up of the viscosity solutions to linear degenerate elliptic equations on the "interior boundaries" in bounded domains in $\mathbb{R}^n$ are given.

Key words: Linear degenerate elliptic equations, viscosity solutions, gradient estimates, gradient blow up

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1. Introduction. Let

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u = f(x) \quad \text{in} \quad \Omega \tag{1}$$

be a second order partial differential equation with non-negative characteristic form (degenerate elliptic equation)

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \ge 0 \quad \text{for} \quad x \in \overline{\Omega}, \xi \in \mathbb{R}^n \setminus \{0\} \tag{2}$$

defined in a bounded region $\Omega \subset \mathbb{R}^n$, $n \ge 2$. In this paper the regularity of viscosity solutions to such equations in a neighbourhood of the so-called "interior boundary" is studied. As it is well known, see paper [1], the correct way to give boundary conditions is determined by means of the so-called Fichera function

$$\beta(x) = \sum_{k=1}^{n} \left(b^k(x) + \sum_{j=1}^{n} a^{kj}_{x_j}(x) \right) \nu_k \quad \text{on} \quad \Gamma, \tag{3}$$

where Γ is the characteristic part of the boundary. Also of interest, however, is the case when Γ is a smooth surface interior to Ω dividing it in two subdomains Ω_1 and Ω_2 so that $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, Γ is characteristic for (1) and, moreover, $\beta(x) = 0$ on Γ. In what follows we assume

$$a^{ij}, b^i, c, f, \psi, \partial\Omega, \Gamma \in C^\infty, \tag{4}$$

$$c(x) \geq c_0 > 0 \quad \text{in} \quad \overline{\Omega}, c_0 = \text{const}. \tag{5}$$

Moreover, if $\nu(x)$ is the unit interior normal to Γ with respect to Ω_1, it is assumed that degeneration occurs in the normal directions $\nu(x)$ only, i.e.

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j = 0 \quad \text{iff} \quad x \in \Gamma \quad \text{and} \quad \xi = \nu(x). \tag{6}$$

In a previous paper [2] we proved that under conditions (2), $\beta(x) = 0$ on Γ and (4)–(6), if the generalized curvature $\mathcal{H}_\Gamma$ (introduced by J. Serrin, see ch. II, sec. 9 of [3] or (13) below) of Γ has constant sign, i.e. $\mathcal{H}_\Gamma \geq 0$ or $\mathcal{H}_\Gamma \leq 0$ for every $x \in \Gamma$, then the following problems:

$$Lu_1 = f \quad \text{in} \quad \Omega_1, \tag{7}$$

$$Lu_2 = f \quad \text{in} \quad \Omega_2, \quad u = \psi \quad \text{on} \quad \partial\Omega \tag{8}$$

have unique viscosity solutions $u_1 \in C(\overline{\Omega}_1)$, $u_2 \in C(\overline{\Omega}_2)$. Moreover, $u_1 = u_2$ on Γ and the function $U(x) = u_1(x)$ in $\overline{\Omega}_1$ and $U(x) = u_2(x)$ in $\overline{\Omega}_2$ is the unique viscosity solution of equation (1) and

$$u = \psi \quad \text{on} \quad \partial\Omega. \tag{9}$$

Assuming that the equation degenerates on Γ, only from the hypoellipticity results in Th. 5.9 in [4] (see also [5, 6]), it follows that $U(x) \in C^\infty(\Omega \setminus \Gamma)$. In [2] we proved that $U(x)$ is Hölder continuous on Γ if $\mathcal{H}_\Gamma = 0$.

The aim of the present paper is to prove Lipschitz regularity or gradient blow up of u_1, u_2 or equivalently U on Γ under sharp conditions. The results are obtained under weaker assumptions then the usual [7].

One motivation to study this problem comes from the optics in the reconstruction of a wave in Fresnel (paraxial) regime from its intensity (see [8] and the references there). The corresponding equation is

$$-\nabla\left(A^2(x,z)\nabla u\right) = \frac{\partial A^2(x,z)}{\partial z} \quad \text{in} \quad \Omega \subset R^2, \tag{10}$$

where $A(x,z)$ is the amplitude of the wave and $u(x)$ is the unknown phase of the wave. Since $A = 0$ on $\partial\Omega$, easy calculations give us that conditions (2), $\beta(x) = 0$ on Γ and (4)–(6) are satisfied in Ω and on $\partial\Omega$.

The same type of degeneracy on some part of the boundary can be observed for the equation of Black–Scholes [9].

Another motivation is the so-called "interior gradient blow up" for the viscosity solutions of fully nonlinear, uniformly elliptic equations, this phenomenon being investigated in Theorem 2.9 and Corollary 2.12 in [10].

The paper is organized as follows: in Section 2 some definitions of viscosity solutions of equation (1) and of boundary value problem (1), (9) are given. Section 3 deals with gradient estimates of viscosity solutions on the "interior boundary" Γ while section 4 is dedicated to gradient blow up of the viscosity solutions.

2. Preliminaries. Throughout this paper we use the notations and definitions from [11] for equation (1) with the notation

$$F(x,u,p,X) = -\mathrm{trace}(A(x)X) + \langle B(x), p\rangle + c(x)u, \tag{11}$$

where $A(x) = \{a_{ij(x)}\}$, $B(x) = (b_i(x))$. Note that (5) is now equivalent to

$$F(x,r,p,X) - F(x,s,p,X) \geq c_0(r-s) \quad \text{for} \quad r \geq s, \tag{12}$$

in $\overline{\Omega}$, $c_0 = \mathrm{const} > 0$.

We rely on the definitions of viscosity sub- and supersolutions from [11], Definition 2.2.

We say that the Dirichlet condition is satisfied in the classical sense, if $u \in C(\bar{\Omega})$ and if $u(x) = \psi(x)$ for every $x \in \partial\Omega$. Unfortunately the classical Dirichlet problem is not stable under small perturbations of the differential equation, see Section 7 in [11]. Therefore one has to weaken condition (9) so it can encompass small perturbations, too. To this end we use Definition 7.4 in [11].

Further on we will use the following comparison principle, Th 3.2 in [2] (see also Ex. 3.6 in [11]).

Proposition 1 (Comparison principle). *Suppose* (2), $\beta(x) = 0$ *on* Γ *and* (4)–(6) *hold. If* $u \in USC(\overline{\Omega})$ $v \in LSC(\overline{\Omega})$ *are viscosity sub- and supersolution of* (1) *and* $u \leq v$ *on* $\partial\Omega$ *then* $u \leq v$ *in* $\overline{\Omega}$.

Let us recall the definition of the generalized mean curvature of some C^2 smooth surfaces introduced by J. Serrin in ch. II., sect. 9 of [3]. If $\lambda^\tau(x)$, $\kappa_\tau(x)$ are the principal directions and the principal curvatures of Γ at some point $x \in \Gamma$ and $\nu(x)$ is the interior unit normal to Γ (with respect to Ω_1), then

$$\mathcal{H}(x) = \lambda^\tau A \lambda^\tau \kappa_\tau + \nu A \nu H \tag{13}$$

is the generalized mean curvature of Γ at the point x. Here $A = \{a^{ij}(x)\}$ and $H = (\kappa_1 + \cdots + \kappa_{n-1})/(n-1)$ is the ordinary mean curvature of Γ.

From (6) we have $\nu A\nu = 0$ on Γ so that for equation (1)

$$\mathcal{H}(x) = \lambda^\tau A\lambda^\tau \kappa_\tau. \tag{14}$$

By means of the Perron method, the following existence and uniqueness result was proved in [2].

Proposition 2 (Th. 4.1 in [2]). *Suppose* (2), $\beta(x) = 0$ *on* Γ *and* (4)–(6) *hold and* $\mathcal{H}(x) \le 0$ *or* $\mathcal{H}(x) \ge 0$ *for every* $x \in \Gamma$. *Then* (7) *has a unique viscosity solution* $u_1(x) \in C(\overline{\Omega}_1)$ *which satisfies equation* (7) *in* $\overline{\Omega}_1$ *without any boundary condition on* $\partial\Omega_1$.

Analogously, (8) has a unique viscosity solution $u_2(x) \in C(\overline{\Omega}_2)$ satisfying equation (8) in $\overline{\Omega}_2\backslash\partial\Omega$ and the Dirichlet condition on $\partial\Omega$ in classical sense.

Finally, $u_1(x) = u_2(x)$ on Γ and the function $U(x) = u_1(x)$ in Ω_1, $U(x) = u_2(x)$ in $\overline{\Omega}_2$ is the unique viscosity solution of (1), (9).

It is proved in Section 5 in [2] that the common value $u_0(x)$ of $u_1(x)$ and $u_2(x)$ on Γ, which plays an important role of the considerations below, is the solution of the uniformly elliptic equation

$$L_0 u = -\sum_{\sigma,\tau=1}^{n-1} A^{\tau\sigma}(x)u_{\lambda_\tau\lambda_\sigma} + \sum_{\tau=1}^{n-1} B^\tau(x)u_{\lambda_\tau} + c(x)u - f(x) = 0 \tag{15}$$

for $x \in \Gamma$. Equation (15) is obtained using a principal coordinate system at $x_0 \in \Gamma$ consisting of the unit normal ν and the principle directions $\lambda^\tau \in T_\Gamma(x_0)$ $(\tau = 1, \ldots, n-1)$. Taking into account (6), we get also

$$a^{ij}(x)\nu_i\nu_j = 0, \quad a^{ik}(x)\nu_i = 0, \quad a^{ij}_{x_s}(x)\nu_i\nu_j = 0 \tag{16}$$

for $k, s = 1, \ldots, n$ and every $x \in \Gamma$ because the function $a^{ij}(x)\xi_i\xi_j$ has a minimum in $\overline{\Omega} \times \mathbb{R}^n$ at the points $x \in \Gamma$, $\xi = \nu(x)$.

Problem (15) has a unique classical solution $u_0(x)$ on C^∞ manifold Γ under conditions (5)–(6).

The following statement is a revised version of Th. 5.1 in [2] treating Hölder continuity of $u_1(x)$ and $u_2(x)$ on Γ. The difference is that the condition $\mathcal{H}_\Gamma = 0$ is replaced with the weaker requirement $\mathcal{H}_\Gamma$ not to change its sign on Γ.

Proposition 3. *Suppose* (2), $\beta(x) = 0$ *on* Γ *and* (4)–(6) *hold and* $\mathcal{H}_\Gamma \ge 0$ *or* $\mathcal{H}_\Gamma \le 0$ *for every* $x \in \Gamma$. *Then the viscosity solutions* $u_1(x) \in C(\overline{\Omega}_1)$ *and* $u_2(x) \in C(\overline{\Omega}_2)$ *of problems* (7) *and* (8) *defined in Prop.* 2.2 *satisfy the boundary data* $u_0(x)$ *on* Γ, *i.e.* $u_1(x) = u_2(x) = u_0(x)$ *on* Γ. *Moreover, the viscosity solution* $U(x) = u_1(x)$ *in* Ω_1, $U(x) = u_2(x)$ *in* $\overline{\Omega}_2$ *of* (1), (9) *is Hölder continuous on* Γ *with exponent* $\lambda \in (0,1)$ *depending on* $\|a^{ij}\|_{C^2(\overline{\Omega})}$, $\|b^i\|_{C^1(\overline{\Omega})}$, c_0 *and* Γ.

Proof. Without loss of generality suppose that $\mathcal{H}_\Gamma \ge 0$. Let $G_i = \Omega_i \cap \{d(x) < d_0\}$, $d(x) = \operatorname{dist}(x, \Gamma)$, where d_0 is sufficiently small so that $d(x) \in C^2(G_i)$, $i = 1, 2$. We consider in G_1 the barrier function $v(x) = u_0(x) + Nd^\lambda(x)$, where $N = \left[\sup_\Gamma |u_0(x)| + \sup_{\overline{\Omega}} |u(x)|\right] d_0^{-\lambda}$ and $u_0(x)$ is extended as a constant along the normals to Γ. If the positive constant λ is chosen sufficiently small, then $v(x)$ is a classical supersolution of (7). Easy calculations in a principle coordinate system give

$$L_1 v \geq -K_2 d + N[-\lambda K_3 + c_0] d^\lambda \geq 0$$

in $\overline{G}_1$ when λ is sufficiently small. Here $f_1(x) = f(x)$ on Γ and is extended as a constant along the normal to Γ, $f_1 \in C^1(\overleftarrow{G}_1)$, $K_2 = \sup_{\overleftarrow{G}_1} |\nabla(f - f_1)|$, $K_3 = \sup_{\overleftarrow{G}_1} |\nabla(b^i \nu_i + \lambda^\tau A \lambda^\tau k_\tau)|$, $A = \{a^{ij}\}$ and λ^τ, k_τ are principle directions and principal curvatures of Γ at the point $y(x) \in \Gamma$, nearest to $x \in G_1$.

Since $\mathcal{H}_\Gamma \geq 0$ from Lemma 4.2 and Remark 4.3 in [2], it follows that $v(x)$ is a viscosity supersolution of (7) in $G_1 \cup \Gamma$, such that $v \geq u_1$ on $\partial G_1 \cap \{d(x) = d_0\}$ and the comparison principle, Lemma 3.4 in [2], yields $v(x) \geq u_1(x)$ in $\overline{G}_1$. In the same way one can prove that $w(x) = u_0(x) - N d^\lambda(x)$ is a viscosity subsolution of (7) and hence $u_2(x) \geq u_0(x) - N d^\lambda(x)$ in $\overline{G}_2$, i.e. $u_0(x) \leq u_2(x) \leq u_1(x) \leq u_0(x)$ for $x \in \Gamma$ and hence $u_1(x) = u_2(x) = u_0(x)$ in Γ. Now by means of the standard comparison principle, Prop. 2.3, one can prove $u_1(x) \geq u_0(x) - N d^\lambda(x)$ in $\overline{G}_1$, $u_2(x) \leq u_0(x) + N d^\lambda(x)$ in $\overline{G}_2$ without curvature conditions for Γ, which gives Prop. 3.

3. Gradient estimates of the viscosity solutions on the "interior boundary". As Prop. 3 shows, the viscosity solutions $u_1(x)$, resp. $u_2(x)$, $U(x)$ are only Hölder continuous functions in $\overline{\Omega}_1$, resp. in $\overline{\Omega}_2$, $\overline{\Omega}$. In $\Omega \backslash \Gamma$ these solutions are C^∞ smooth (see Th. 5.9 in [4] or [5]) and the open question is their higher regularity in a neighbourhood of Γ.

Let us mention that a sufficient condition for gradient estimates of the generalized $L^p(\Omega)$ solutions of (1), (9) is proved in ([7], Th. 1.8.1) by means of elliptic regularization. The main condition is $\sup_{x \in E_\delta} B_1(x) > 0$, where E_δ is a neighbourhood of the set E of degeneracy of L, i.e. $E = \{x \in \Omega; \det\{a^{ij}(x)\} = 0\}$,

$$B_1(x) = c(x) - \frac{1}{4} M n + \max_s \left\{ b^s_{x_s} - \frac{1}{2} \sum_{j=1}^n |b^j_{x_s}| - \frac{1}{2} \sum_{j=1, j\neq s}^n |b^s_{x_j}| \right\}$$

and $M = \min_{1 \leq s \leq n} M_s$ for

$$M_s = \inf_{v \in W^{2,\infty}(\Omega)} \inf_{x \in \overline{E}_\delta} \left(\sum_{k,j=1}^n a^{kj}_{x_s} v_{x_k x_j} \right)^2 \Big/ \left(\sum_{i,j,k,s=1}^n a^{kj} v_{x_k x_s} v_{x_j x_s} \right).$$

We will give precise conditions for gradient estimates or gradient blow up of the viscosity solutions on Γ.

Theorem 1. *Suppose* (2), $\beta(x) = 0$ *on* Γ *and* (4)–(6) *hold and* $\mathcal{H}_\Gamma \geq 0$ *or* $\mathcal{H}_\Gamma \leq 0$ *for every* $x \in \Gamma$. *If*

$$\frac{\partial b^i}{\partial \nu}(x) \nu_i + \lambda^\tau \frac{\partial A}{\partial \nu}(x) \lambda^\tau \kappa^\tau + \lambda^\tau A(x) \lambda^\tau \kappa_\tau^2 + c(x) \geq c_1, \tag{17}$$

$c_1 = \text{const} > 0$ *for every* $x \in \Gamma$ *then the viscosity solutions* $u_1(x) \in C(\overline{\Omega}_1)$ *and* $u_2(x) \in C(\overline{\Omega}_2)$ *of problems* (7) *and* (8) *defined in Prop.* 2 *are Lipschitz continuous in the neighbourhood of* Γ.

Sketch of the proof. Using the same technique as in obtaining (15) and (16), in $G_i = \Omega_i \cap \{d(x) < d_0\}$ equation (1) can be rewritten in the form

$$\begin{aligned}(18)\quad Lu = -A^{nn}(x)u_{\nu\nu} - A^{\tau\sigma}(x)u_{\lambda^\tau\lambda^\sigma} - A^{\tau n}(x)u_{\lambda^\tau\nu} - A^{n\tau}(x)u_{\nu\lambda^\tau}\\ + B^n(x)u_\nu + B^\tau(x)u_{\lambda^\tau} + c(x)u - f(x) = 0.\end{aligned}$$

Moreover, from (3), (16) and the derivatives of (16) in λ^τ directions on Γ, we get

$$0 = \beta(x) = b^i(x)\nu_i + \mathcal{H}, \tag{19}$$

$$A^{nn}(x) = A^{\tau n}(x) = A^{n\tau}(x) = B^n(x) = 0 \tag{20}$$

for every $x \in \Gamma$ and $\tau = 1, \ldots, n-1$. Let us consider in G_1 the barrier function $v(x) = u_0(x) + Nd(x)$. Here the constant N will be chosen sufficiently large and the classical solution $u_0(x)$ of (15) is extended as a constant along the normals to Γ in G_i. As in the proof of Prop 2.5. we get

$$L_1 v \geq -K_1 d + N\left[b^i(x)\nu_i + \lambda^\tau A(x)\lambda^\tau k_\tau + \lambda^\tau A(x)\lambda^\tau k_\tau^2 d + c(x)d - K_2 d^2\right]. \tag{21}$$

Here $f_1(x) = f(x)$ on Γ and $f_1(x)$ is extended as a constant along the normals to Γ in G_i, $f_1 \in C^1(\overline{G}_i)$, $K_1 = \sup_{\overline{G}_i} |\nabla(f - f_1)|$, $K_2 = \sup_{\overline{G}_i} |\frac{\lambda^\tau A(x)\lambda^\tau k_\tau^3}{1-k_\tau d}|$. When N is sufficiently large and d_0 is sufficiently small, we get $L_1 v \geq 0$ in G_1, $v = u_1$ on Γ, $v \geq u_1$ for every $x \in \partial G_1 \cap \{d(x) = d_0\}$ and from the comparison principle, Proposition 1, it follows that $u_1(x) \leq v(x) = u_0(x) + Nd(x)$ for every $x \in \overline{G}_1$. In the same way one can prove that the function $w(x) = u_0(x) - Nd(x)$ is a viscosity subsolution of (7) in G_1, i.e.

$$u_0(x) - Nd(x) \leq u_1(x) \leq u_0(x) + Nd(x) \quad \text{for every} \quad x \in \overline{G}_1. \tag{22}$$

Inequalities (22) prove Theorem 1 for $u_1(x)$. The proof of the Lipschitz continuity of $u_2(x)$ is similar and is omitted here.

4. Gradient blow up of the viscosity solutions on the "interior boundary". In this section we will show that condition (17) in Theorem 3.1 is sharp.

Theorem 2. *Suppose* (2), $\beta(x) = 0$ *on* Γ *and* (4)–(6) *hold and* $\mathcal{H}_\Gamma \geq 0$ *or* $\mathcal{H}_\Gamma \leq 0$ *for every* $x \in \Gamma$. *If*

$$\frac{\partial b^i}{\partial \nu}(x_0)\nu_i + \lambda^\tau \frac{\partial A}{\partial \nu}(x_0)\lambda^\tau \kappa^\tau + \lambda^\tau A(x_0)\lambda^\tau \kappa_\tau^2 + c(x_0) < 0 \tag{23}$$

for some $x_0 \in \Gamma$,

$$f(x) \equiv 0 \quad \text{on} \quad \Gamma, f(x) \not\equiv 0 \quad \text{in} \quad \Omega_1 \tag{24}$$

and either

$$f(x) \le 0 \quad \text{in} \quad \Omega_2 \quad \text{or} \quad f(x) \ge 0 \quad \text{in} \quad \Omega_2, \tag{25}$$

then the gradient of the viscosity solution $u_1(x_0)$ *blows up at* x_0 *in direction* $\nu(x_0)$ *where* $\nu(x_0)$ *is the interior normal to* Γ *at* x_0 *with respect to* Ω_1. *More precisely*

$$\lim_{t\to+0} \frac{u_1(x_0 + t\nu(x_0)) - u_1(x_0)}{t} = -\infty \quad \text{or} \quad \lim_{t\to+0} \frac{u_1(x_0 + t\nu(x_0)) - u_1(x_0)}{t} = \infty \tag{26}$$

respectively.

Sketch of the proof. Since $f(x) \equiv 0$ on Γ the unique solution of (15) is $u_0 = 0$. Suppose that (25) holds. Then the function $\overline{u} = 0$ is a classical supersolution of (7) in Ω_1 because

$$L_1\overline{u} = -f \ge 0 = L_1 u_1$$

in Ω_1 and $\overline{u} = u_1$ on Γ. From the comparison principle, Prop. 1, it follows that $u_1 \le \overline{u} = 0$ in $\overline{\Omega}_1$. If $u_1(y) = 0$ at some interior point $y \in \Omega_1$, then from the strong interior maximum principle it follows that $u_1 \equiv 0$ on every compact $K \subset \Omega_1$, $y \in K$, and hence $f \equiv 0$ in K contrary to (24). Note that from the hypoellipticity of equation (1) in Ω_1 we have $u_1 \in C^\infty(\Omega_1)$. Hence

$$u_1(x) < 0 \quad \text{in} \quad \Omega_1, \quad u_1(x) = 0 \quad \text{on} \quad \Gamma. \tag{27}$$

From (23) there exist a constant $c_2 > 0$ and a ball $B_0(x_0, r_0) = \{|x - x_0| < r_0\}$ such that

$$\frac{\partial b^i}{\partial \nu}(x)\nu_i + \lambda^\tau A_\nu(x)\lambda^\tau \kappa_\tau^2 + \lambda^\tau \frac{\partial A}{\partial \nu}(x)\lambda^\tau \kappa_\tau + c(x) \le -2c_2$$

for every $x \in B_0(x_0, r_0) \cap \overline{\Omega}_1$. Here the normal ν and the principle directions λ^τ are calculated at the point $y(x) \in \Gamma$ nearest to x.

Let Γ_1 be an internal touching surface to $\partial\Omega_1$ at the point $x \in \Gamma$, $\Gamma_1 \subset \Omega_1$, $\Gamma_1 \cap \Gamma = \{x_0\}$. We consider a domain $G \subset \Omega_1$ with boundary $\partial G = \Gamma_1 \cup \Gamma_0$, $\Gamma_0 \subset \partial B_0$. If n, μ^τ, k_τ are the unite interior normal to Γ_1 (with respect to G) and the principle directions and the principle curvatures of Γ_1, we suppose that Γ_1 is sufficiently close to Γ and r_0 is sufficiently small so that $\lambda^\tau(x_0) = \mu^\tau(x_0)$, $\kappa_\tau(x_0) = k_\tau(x_0)$ and the inequality

$$\frac{\partial b^i}{\partial n}(x)n_i + \mu^\tau A(x)\mu^\tau k_\tau^2 + \mu^\tau \frac{\partial A}{\partial n}(x)\mu^\tau k_\tau + c(x) \le -c_2 \tag{28}$$

holds for every $x \in G$. Here n, μ^τ, k_τ are calculated at the point $y(x) \in \Gamma_1$ nearest to x.

Let $\rho(x) = \mathrm{dist}(x, \Gamma_1)$, $\Gamma_0 = \partial G \cap \partial B_0$ and let us define $\sup_{x\in\Gamma_0} u_1(x) = -M$, $K_1 = \sup_{\overline{G}} \frac{nA(x)n}{\rho^2(x)}$, $K_2 = \sup_{\overline{G}} |\frac{\mu^\tau A(x)\mu^\tau k_\tau^3}{1-k_\tau d}|$. Note that $M > 0$ from (27) while $K_1 < \infty$ from (16) and the choice of Γ_1.

We consider the barrier function $v = -\epsilon\rho^\lambda(x)$ in G, where $\sup_{\overline{G}} \rho(x) \le \frac{C_2}{4K_2}$ is fixed, ϵ is sufficiently small and λ is sufficiently close to 1.

Repeating the calculations in the proof of Proposition 3 we get

$$L_1 v \ge \epsilon\rho^\lambda \left[-\frac{C_2}{4} + \frac{C_2}{2} - \frac{C_2}{4} \right] = 0. \tag{29}$$

Since $v \ge -\epsilon \sup_{x\in\Gamma_1} \rho^\lambda(x) = -M \ge u_1$ for $x \in \Gamma_1$ and $v = 0 \ge u_1$ for $x \in \Gamma_0$q from the comparison principle, Proposition 1, it follows that

$$-\epsilon\rho^\lambda(x) = v(x) \ge u_1(x) \quad \text{for every} \quad x \in G. \tag{30}$$

Hence for every $x = x_0 + \rho n(x_0) = x_0 + \rho(x)\nu(x_0)$ we get from (30)

$$\lim_{t\to+0} \frac{u_1(x_0 + t\nu(x_0)) - u_1(x_0)}{t} = \lim_{t\to+0} \frac{u_1(x_0 + t\nu(x_0))}{t} \le -\epsilon \lim_{t\to+0} \frac{t^\lambda}{t} = -\infty$$

which proves (26).

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