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OUTPUT FEEDBACK STABILIZATION OF AN ANAEROBIC DIGESTION MODEL

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Abstract

We study a nonlinear model of a biological wastewater treatment process, involving two microbial populations and two substrates and producing biogas (methane). A nonadaptive feedback control law for asymptotic stabilization of the closed-loop system towards a previously chosen operating point is proposed. An extremum seeking algorithm is applied to stabilize the system dynamics towards the maximum methane flow rate. Computer simulations are reported to illustrate the robustness of the feedback under model uncertainties.
Key words: biological wastewater treatment, nonlinear dynamic model, feedback control, asymptotic stabilization, model-based extremum seeking
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1. Introduction. Anaerobic digestion is a process where microorganisms decompose the organic compounds into biogas (a gas mixture of methane and carbon dioxide). The process can also be adapted to wastewater treatment, because it reduces the chemical oxygen demand (COD) in the outlet stream below a specified value, usually fixed by environmental and safety rules, see e. g. [2,7,10].

In this paper we consider a well known model of an anaerobic digestion process, based on two main reactions [2,7,8]: (a) acidogenesis, where the organic substrate (denoted by s_1) is degraded into volatile fatty acids (VFA, denoted by s_2) by acidogenic bacteria (x_1); (b) methanogenesis, where VFA are degraded into methane CH_4 and carbon dioxide CO_2 by methanogenic bacteria (x_2). The mass balance model in a continuously stirred bioreactor is described by the following nonlinear system of ordinary differential equations:

$$
(1)\qquad \begin{aligned}
\dot{s}_1 &= u(s_1^i - s_1) - k_1\,\mu_1(s_1)\,x_1,\\
\dot{x}_1 &= (\mu_1(s_1) - \alpha u)x_1,\\
\dot{s}_2 &= u(s_2^i - s_2) + k_2\,\mu_1(s_1)x_1 - k_3\,\mu_2(s_2)\,x_2,\\
\dot{x}_2 &= (\mu_2(s_2) - \alpha u)x_2
\end{aligned}
$$

with output

$$
(2)\qquad Q = k_4\,\mu_2(s_2)\,x_2.
$$

The definitions of the model variables and parameters are given in Table 1. There the constants m_1, m_2, k_{s_1}, k_{s_2}, k_I are related to the particular expressions of the specific growth rate functions μ_1 and μ_2, used later in Section 4.

Table 1

Definition of the model parameters

s_1	concentration of chemical oxygen demand (COD) [g/l]
s_2	concentration of volatile fatty acids (VFA) [mmol/l]
x_1	concentration of acidogenic bacteria [g/l]
x_2	concentration of methanogenic bacteria [g/l]
u	dilution rate [day^{-1}]
s_1^i	influent concentration s_1 [g/l]
s_2^i	influent concentration s_2 [mmol/l]
k_1	yield coefficient for COD degradation [g COD/(g x_1)]
k_2	yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	yield coefficient for VFA consumption [mmol VFA/(g x_2)]
k_4	coefficient [l^2/g]
m_1	maximum acidogenic biomass growth rate [day^{-1}]
m_2	maximum methanogenic biomass growth rate [day^{-1}]
k_{s_1}	saturation parameter associated with s_1 [g COD/l]
k_{s_2}	saturation parameter associated with s_2 [mmol VFA/l]
k_I	inhibition constant associated with s_2 [(mmol VFA/l)$^{1/2}$]
α	proportion of dilution rate reflecting process heterogeneity
Q	methane gas flow rate

Practical experiments show that adaptive feedback control is a very appropriate tool for asymptotic stabilization in the case of model uncertainties. Such a feedback for this model is proposed and studied by the authors in [5,6]. There, following [2,10], the dynamic model (1) is extended by adding an additional (auxiliary) differential equation of the form

$$
(3)\qquad \dot{\beta} = -C(\beta - \beta^-)(\beta^+ - \beta)k_4\,\mu_2(s_2)\,x_2\,(s - \bar{s}),
$$

where the positive constants C, β^- and β^+ are chosen in a proper way; $\bar{s}$ is a previously chosen operating point representing the biological oxygen demand, i. e. $\bar{s} = \dfrac{k_2}{k_1} s_1 + s_2$ and the latter is assumed to be on-line measurable. The solution $\beta = \beta(t)$ of (3) is used to define a feedback control low

$$k(\cdot) = \beta k_4 \mu_2(s_2) x_2. \tag{4}$$

Depending on $\bar{s}$, an equilibrium point $\bar{p}$ of the system (1), (3) is computed and it is proved that the feedback (4) stabilizes asymptotically the dynamics towards $\bar{p}$ [5,6]. Moreover, choosing in a proper way different operating points $\bar{s}$, it is possible to drive the dynamics towards the equilibrium point, where the maximum methane flow rate Q is achieved. This is realized by means of a numerical extremum seeking algorithm. Optimization via extremum (peek) seeking is recently an extensively used approach in optimizing the performance of a continuously stirred bioreactor. The usual and known extremum seeking method [4] is not model-based; it is presented in the form of a diagram (scheme) to adjust the dilution rate directly in the bioreactor in order to steer the process to a point, where optimal value of the output is achieved. The main restriction in applying this model-free extremum seeking approach is that the dynamics should be open-loop stable; otherwise, a locally stabilizing controller is needed. Our approach, in turn, is different: we first globally stabilize the dynamics towards an equilibrium point and then apply the numerical extremum seeking algorithm to drive the system to the desired state.

As it is mentioned above, the adaptive output feedback (4) uses the solution of the auxiliary differential equation (3). Thus instead of a four-dimensional dynamical model we have to investigate a five-dimensional system, which is more complicated; moreover, the additional equation (3) cannot be interpreted in terms of the process dynamics. The question is whether it is possible to avoid the auxiliary equation, i. e. whether we can choose in a proper way a constant $\beta > 0$ and use a feedback of the form (4) to globally stabilize the dynamics (1) to a desired state. Here we give positive answer to this question.

2. Global asymptotic stabilization of the model. Consider the control system (1) in the state space $\zeta = (s_1, x_1, s_2, x_2)$.

In what follows we assume that the input substrate concentrations s_1^i and s_2^i are constant and the methane flow rate Q is the measurable output. The dilution rate u is considered as a control input. The functions μ_1 and μ_2 model the bacterial specific growth rates. We do not assume to know explicit expressions for the latter, we only impose the following general assumptions on μ_1 and μ_2:

Assumption A1. $\mu_j(s_j)$ is defined for $s_j \in [0, +\infty)$, $\mu_j(0) = 0$, $\mu_j(s_j) > 0$ for $s_j > 0$; $\mu_j(s_j)$ is continuously differentiable and bounded for all $s_j \in [0, +\infty)$, $j = 1, 2$.

Define $s^i = \dfrac{k_2}{k_1} s_1^i + s_2^i$ and let the following assumption be satisfied:

Assumption A2. Lower bounds s^{i-} and k_4^- for the values of s^i and k_4, as well as an upper bound k_3^+ for k_3 are known.

Define the following feedback control law:

$$k(\zeta) = \beta k_4 \mu_2(s_2) x_2 \quad \text{with} \quad \beta \in \left(\frac{k_3^+}{s^{i-} k_4^-}, +\infty \right). \tag{5}$$

Taking into account the expression for Q (see (2)), the feedback k depends only on the parameter β and the measurable output Q.

Choose some $\beta \in \left(\dfrac{k_3^+}{s^{i-} k_4^-}, +\infty \right)$ and define a reference point

$$\bar{s} = s^i - \frac{k_3}{\beta k_4}. \tag{6}$$

Obviously, $\bar{s}$ belongs to the interval $(0, s^i)$.

Assumption A3 (regulability [7]). There exists a point $\bar{s}_1$ such that

$$\mu_1(\bar{s}_1) = \mu_2 \left(\bar{s} - \frac{k_2}{k_1} \bar{s}_1 \right) > 0, \quad \bar{s}_1 \in \left(0, s_1^i\right).$$

Denote by Σ the closed-loop system obtained from (1) by substituting the control variable u by the feedback $k(\zeta)$ from (5).

Find $\bar{s}_1$ according to Assumption A3 and define the points

$$\bar{s}_2 = \bar{s} - \frac{k_2}{k_1} \bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad \bar{x}_2 = \frac{1}{\alpha \, \beta \, k_4}. \tag{7}$$

It is straightforward to see that $\bar{\zeta} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ is an equilibrium point for the closed-loop system Σ. We shall prove below that the feedback law (5) asymptotically stabilizes Σ to $\bar{\zeta}$. Denote $s = \dfrac{k_2}{k_1} s_1 + s_2$ and define the sets

$$\begin{aligned}
\Omega_0 &= \left\{ (s_1, x_1, s_2, x_2) |\ s_1 > 0,\ x_1 > 0,\ s_2 > 0,\ x_2 > 0 \right\}, \\
\Omega_1 &= \left\{ (s_1, x_1, s_2, x_2) |\ s_1 + k_1 x_1 \le \frac{s_1^i}{\alpha},\ s + k_3 x_2 \le \frac{s^i}{\alpha} \right\}, \\
\Omega_2 &= \left\{ \left(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2 \right) \mid 0 < s_1 < \frac{k_1}{k_2} \bar{s},\ x_1 > 0 \right\}, \\
\Omega &= \Omega_0 \cap \Omega_1.
\end{aligned}$$

Assumption A4. Let the inequality $\mu_1'(s_1) + \frac{k_2}{k_1} \cdot \mu_2'\left(\bar{s} - \frac{k_2}{k_1} s_1\right) > 0$ be satisfied for each $s_1 \in \left(0, \frac{k_1}{k_2}\bar{s}\right)$.

Assumption A4 is technical and it is used in the proof of Theorem 1 below. It will be discussed in more details later in Section 4, where the bacterial growth rates μ_1 and μ_2 are specified as the Monod and Haldane law.

Theorem 1. *Fix an arbitrary number* $\beta \in \left(\frac{k_3^+}{s^{i-} \cdot k_4^-}, +\infty\right)$ *and let* $\bar{\zeta} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ *be the corresponding equilibrium point. Let the assumptions* A1, A2, A3 *and* A4 *be satisfied. Then the feedback control law* $k(\cdot)$ *defined by* (5) *stabilizes asymptotically the control system* (1) *to the point* $\bar{\zeta}$ *for any starting point* ζ_0 *from* Ω_0.

Proof. It is straightforward to see that for any starting point $\zeta_0 \in \Omega_0$ and any positive value $u_0 > 0$ for the control, the corresponding trajectory enters the set Ω after a finite time and remains in Ω (cf. [5,6]). For that reason we shall consider the control system (1) only on the set Ω.

Using the definitions of s and $k(\cdot)$, it is easy to see that the following ordinary differential equations are satisfied:

$$\begin{aligned} \dot{s} &= -k(\zeta) \cdot (s - \bar{s}), \\ \dot{x}_2 &= -\alpha k(\zeta) \cdot (x_2 - \bar{x}_2). \end{aligned} \tag{8}$$

Integrating the equations (8) we obtain

$$\begin{aligned} s(t) &= \bar{s} + (s(0) - \bar{s}) \cdot e^{-\int_0^t \beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau}, \\ x_2(t) &= \bar{x}_2 + (x_2(0) - \bar{x}_2) \cdot e^{-\int_0^t \alpha\beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau}. \end{aligned}$$

Since the integrands are strictly positive, we have that for all $t > 0$ the following inequalities hold true:

$$\begin{aligned} &\max\{\bar{s}, s(0)\} \geq s(t) \geq \min\{\bar{s}, s(0)\} > 0, \\ &\max\{\bar{x}_2, x_2(0)\} \geq x_2(t) \geq \min\{\bar{x}_2, x_2(0)\} > 0. \end{aligned} \tag{9}$$

For each point $\zeta = (s_1, x_1, s_2, x_2) \in \Omega$ define the function

$$V(\zeta) = (s - \bar{s})^2 + (x_2 - \bar{x}_2)^2;$$

clearly, the values of this function are nonnegative. Denote by $\dot{V}(\zeta)$ the Lie derivative of V with respect to the right-hand side of (8); then for each $\zeta \in \Omega$,

$$\dot{V}(\zeta) = -k(\zeta) \cdot \left((s - \bar{s})^2 + \alpha\,(x_2 - \bar{x}_2)^2\right) \leq 0.$$

Applying LaSalle's invariance principle (cf. [9]), it follows that every solution of Σ starting from a point in Ω is defined in the interval $[0,+\infty)$ and approaches the largest invariant set with respect to Σ, which is contained in the set $\widetilde{\Omega}$, where $\widetilde{\Omega}$ is the closure of the set $\Omega \cap \Omega_2$. Taking into account the presentation $\bar{s} = \dfrac{k_2}{k_1}\bar{s}_1 + \bar{s}_2$ and $s_1^i = \bar{s}_1 + \alpha k_1 \bar{x}_1$ (see (7)), it can be directly checked that the dynamics of Σ on $\widetilde{\Omega}$ is described by

$$
\begin{aligned}
\dot{s}_1 &= -\frac{1}{\alpha}\,\chi(s_1)\,\left(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1)\right) - k_1\left(\mu_1(s_1) - \chi(s_1)\right)\,x_1, \\
\dot{x}_1 &= (\mu_1(s_1) - \chi(s_1))x_1,
\end{aligned} \tag{10}
$$

where $\chi(s_1) = \mu_2\left(\bar{s} - \dfrac{k_2}{k_1}s_1\right)$. The proof follows further the same idea from Theorem 1 in [6]; we sketch it here for completeness. Consider the function

$$W(s_1, x_1) = (s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 + \alpha(1-\alpha)k_1^2(x_1 - \bar{x}_1)^2.$$

Obviously, $W(s_1, x_1) \geq 0$; moreover, for each point $\left(s_1, x_1, \bar{s} - \dfrac{k_2}{k_1}s_1, \bar{x}_2\right) \in \widetilde{\Omega}$, the Lie derivative $\dot{W}(s_1, x_1)$ along the trajectories of (10) satisfies the inequality $\dot{W}(s_1, x_1) \leq 0$. Let $L^+(\zeta_0)$ be the ω-limit set of any solution $\phi(t, \zeta_0)$ of Σ starting from $\zeta_0 \in \Omega$. The invariance of Ω with respect to the trajectories of Σ implies that $L^+(\zeta_0)$ is a nonempty compact connected invariant set which is a subset of $\widetilde{\Omega}$. Using an extension of LaSalle's invariance principle (Theorem 6 in [3]) it follows that $L^+(\zeta_0)$ is contained in one connected component of the set $\left\{(s_1, x_1, \bar{s} - \dfrac{k_2}{k_1}s_1, \bar{x}_2) \in \widetilde{\Omega} :\ \dot{W}(s_1, x_1, \bar{s} - \dfrac{k_2}{k_1}s_1, \bar{x}_2) = 0\right\}$. Further, Lemma 1 from [7] implies the existence of $x_1^{\min} > 0$ and $s_1^{\min} > 0$ such that $x_1(t) \geq x_1^{\min} > 0$ and $s_1(t) \geq s_1^{\min} > 0$ for each $t > 0$. This leads to $L^+(\zeta_0) = \{\bar{\zeta}\}$. The proof is completed by the fact that the point $\bar{\zeta}$ is a Lyapunov stable equilibrium for the closed-loop system Σ, see [6]. □

3. Model-based extremum seeking. Denote by β some constant from the interval $\left(\dfrac{k_3^+}{s^{i-} \cdot k_4^-}, +\infty\right)$, define the reference point $\bar{s} = \bar{s}_\beta$ (see (6)) and consider the equilibrium point $\bar{\zeta}_\beta = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ where $\bar{s}_1$, $\bar{x}_1$, $\bar{s}_2$ and $\bar{x}_2$ are computed according to (7). Here and in the following we explicitly emphasize that $\bar{s}$ as well as $\bar{\zeta}$ depend on the parameter β. Let the assumptions A1, A2, A3 and A4 hold true. Assume further that the output $Q(\bar{\zeta}_\beta) = k_4\,\mu_2(\bar{s}_2)\,\bar{x}_2$, defined on the set of all steady states $\bar{\zeta}_\beta$ has a maximum at a unique equilibrium point $\zeta_{\beta_*} = (s_1^*,\, x_1^*,\, s_2^*,\, x_2^*)$, i. e. $Q_{\max} = Q(\zeta_{\beta_*})$. Our goal now is to stabilize the dynamics towards the maximum methane flow rate $Q_{\max}$. Obviously, the feedback control law $k(\cdot)$ from (5) can be represented as

$$k_\beta(t) = \beta \cdot Q(t). \tag{11}$$

According to Theorem 1, this feedback will asymptotically stabilize the closed-loop system Σ to the point $\bar{\zeta}_\beta$.

To stabilize the closed-loop system Σ towards $Q_{\max}$ we use an iterative model-based extremum seeking algorithm. The algorithm is presented in details in [5] for the case of the adaptive feedback stabilization, i. e. for the dynamics (1), (3) using the control law (4). Now the algorithm is adapted for the case of nonadaptive (state) stabilization considered here. The main idea of the algorithm is based on the fact that Theorem 1 is valid for *any value* of $\beta > \dfrac{k_3^+}{s^{i-} \cdot k_4^-}$. Thus we can construct a sequence of points $\beta^1, \beta^2, \dots, \beta^n, \dots$, converging to β_*, and generate in a proper way a sequence of values $Q^1, Q^2, \dots, Q^n, \dots$ which converges to $Q_{\max}$. The algorithm is carried out in two stages: on Stage I, an interval $[\beta] = [\beta^-, \beta^+]$ is found such that $[\beta] > \dfrac{k_3^+}{s^{i-} \cdot k_4^-}$ and $\beta_* \in [\beta]$; on Stage II, the interval $[\beta]$ is refined using an elimination procedure based on a Fibonacci search technique. Stage II produces the final interval $[\beta_*] = [\beta_*^-, \beta_*^+]$ such that $\beta_* \in [\beta_*]$ and $\beta_*^+ - \beta_*^- \le \varepsilon$, where the tolerance $\varepsilon > 0$ is specified by the user.

4. Numerical simulation. In the computer simulation, we consider for $\mu_1(s_1)$ and $\mu_2(s_2)$ the Monod and the Haldane model functions for the specific growth rates, which are used in the original model [1,7,8]

$$\mu_1(s_1) = \frac{m_1 s_1}{k_{s_1} + s_1}, \quad \mu_2(s_2) = \frac{m_2 s_2}{k_{s_2} + s_2 + \left(\frac{s_2}{k_I}\right)^2}. \tag{12}$$

Obviously, $\mu_1(s_1)$ and $\mu_2(s_2)$ satisfy Assumption A1: $\mu_1(s_1) < m_1$, $\mu_2(s_2)$ takes its maximum at the point $s_2^m = k_I\sqrt{k_{s_2}}$. Simple derivative calculations imply that if $\bar{s}_\beta$ is chosen such that $0 < \bar{s}_\beta < s_2^m$, then $\mu_2'\left(\bar{s}_\beta - \dfrac{k_2}{k_1}s_1\right) > 0$ holds true, thus Assumption A4 is satisfied. Moreover, if the point $\bar{s}_\beta$ is sufficiently small, then Assumptions A3 and A4 are simultaneously satisfied.

Usually the formulation of the growth rates is based on experimental results, and therefore it is not possible to have exact analytic forms of these functions, but only some quantitative bounds. Assume that we know bounds for $\mu_1(s_1)$ and $\mu_2(s_2)$, i. e. $\mu_j(s_j) \in [\mu_j(s_j)] = [\mu_j^-(s_j), \mu_j^+(s_j)]$ for all $s_j \ge 0$, $j = 1, 2$. This uncertainty can be simulated by assuming in (12) that instead of exact values for the coefficients m_1, k_{s_1}, m_2, k_{s_2} and k_I we have compact intervals for them

$$m_1 \in [m_1] = [m_1^-, m_1^+], \quad k_{s_1} \in [k_{s_1}] = [k_{s_1}^-, k_{s_1}^+],$$

$$m_2 \in [m_2] = [m_2^-, m_2^+], \quad k_{s_2} \in [k_{s_2}] = [k_{s_2}^-, k_{s_2}^+], \quad k_I \in [k_I] = [k_I^-, k_I^+].$$

Then the boundary functions for μ_1 and μ_2 can be computed by

$$\begin{aligned}\mu_1^-(s_1) &= \frac{m_1^- s_1}{k_{s_1}^+ + s_1}, \quad \mu_1^+(s_1) = \frac{m_1^+ s_1}{k_{s_1}^- + s_1},\\ \mu_2^-(s_2) &= \frac{m_2^- s_2}{k_{s_2}^+ + s_2 + \left(\frac{s_2}{k_I^-}\right)^2}, \quad \mu_2^+(s_2) = \frac{m_2^+ s_2}{k_{s_2}^- + s_2 + \left(\frac{s_2}{k_I^+}\right)^2}.\end{aligned}$$

Any $\mu_j(s_j) \in [\mu_j(s_j)]$, $j = 1, 2$, satisfies Assumption A1. There exist also intervals for the kinetic coefficients, such that Assumption A4 is satisfied for any $\mu_j(s_j) \in [\mu_j(s_j)]$, $j = 1, 2$; such intervals are for example the following:

$$\begin{aligned}[m_1] &= [1.2,\ 1.4], \quad [k_{s_1}] = [6.5,\ 7.2],\\ [m_2] &= [0.64,\ 0.84], \quad [k_{s_2}] = [9,\ 10.28], \quad [k_I] = [15,\ 17].\end{aligned}$$

To simulate Assumption A2 we assume intervals for the coefficients k_j to be given, i. e. $k_j \in [k_j] = [k_j^-, k_j^+]$, $j = 1, 2, 3, 4$; such numerical intervals are

$$[k_1] = [9.5, 11.5], \quad [k_2] = [27.6, 29.6], \quad [k_3] = [1064, 1084], \quad [k_4] = [650, 700].$$

All above intervals are chosen to enclose the numerical coefficient values derived by experimental measurements [1]; the values $\alpha = 0.5$, $s_1^i = 7.5$, $s_2^i = 75$ are also taken from [1].

In the simulation process we proceed in the following way. At the initial time $t_0 = 0$ we take random values for the coefficients from the corresponding intervals. We apply the extremum seeking algorithm to stabilize the system towards $Q_{\max}$. Then, at some time $t_1 > t_0$, we choose another set of random coefficient values and

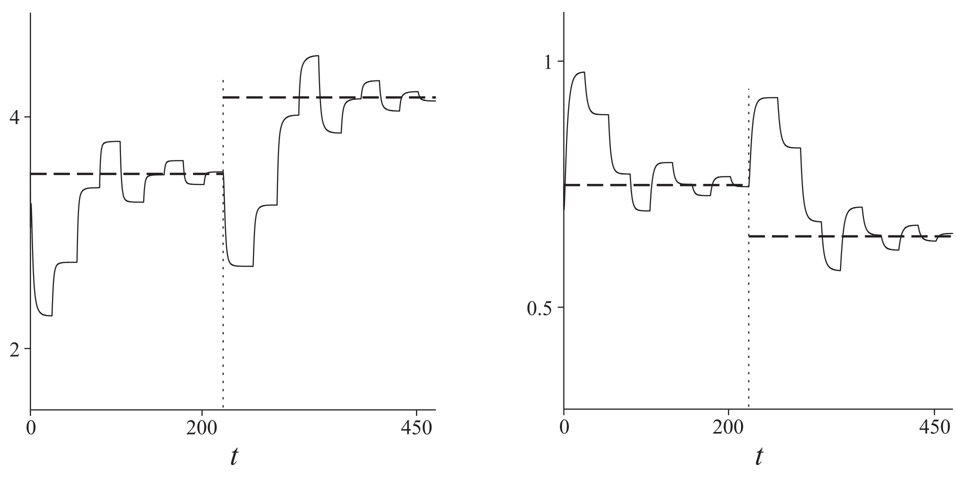

Fig. 1. Time evolution of the phase variables $s_1(t)$ (left) and $x_1(t)$ (right)

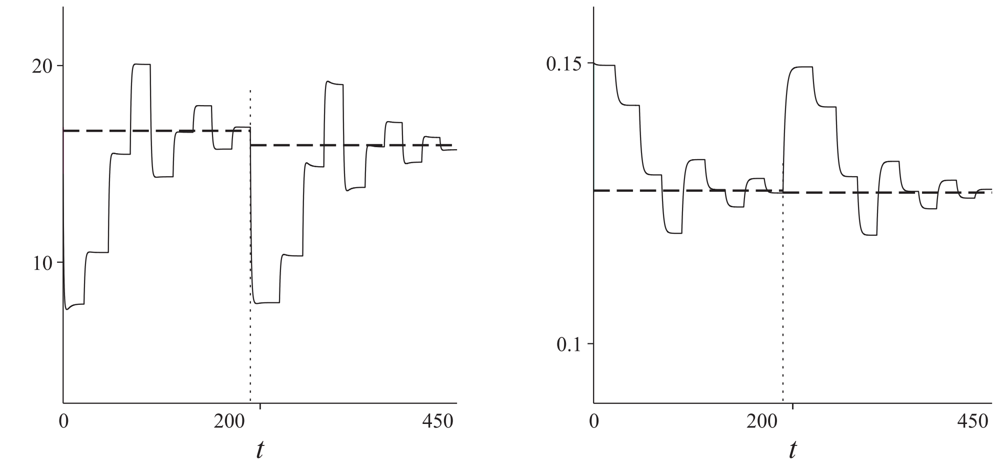

Fig. 2. Time evolution of the phase variables $s_2(t)$ (left) and $x_2(t)$ (right)

repeat the process. Figures 1, 2 and 3 show time profiles of the phase variables $s_1(t)$ and $x_1(t)$, $s_2(t)$ and $x_2(t)$, as well as of $Q(t)$ and of the feedback $k_\beta(t)$ from (11) respectively; there the vertical dot-lines mark the time moment t_1, when the new coefficient values are taken in a random way from the corresponding intervals. The horizontal dash-lines pass through the equilibrium components s_1^*, x_1^*, s_2^*, x_2^* where $Q_{\max}$ is achieved. The “jumps” in the graphs correspond to the different choices of β by executing the algorithmic steps.

5. Conclusion. The paper is devoted to the nonadaptive asymptotic stabilization of a four-dimensional nonlinear dynamic system, which models anaerobic

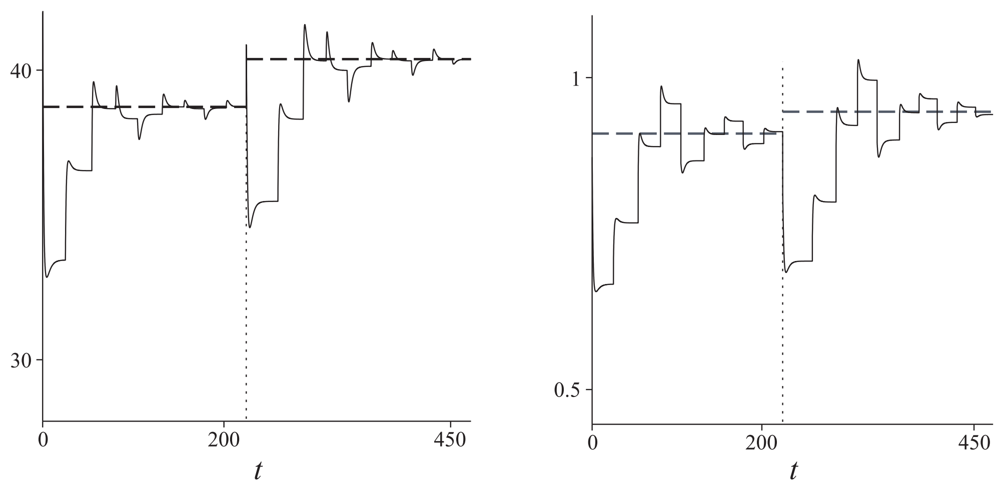

Fig. 3. Time evolution of the output $Q(t)$ (left) and the feedback $k_\beta(t)$ (right)

degradation of organic wastes and produces biogas. The asymptotic stabilization of the dynamic system is realized by means of a nonlinear output (state) feedback control law. The controller uses only the available measurement which is the output methane flow rate. The feedback depends on a parameter β (called feedback gain), which lower bound is assumed to be known. It is shown in Theorem 1 that for any admissible value of β, the system can be asymptotically stabilized to an equilibrium (operating) point $\bar{\zeta}_{\beta}$. The proof of Theorem 1 is very similar to the proof of the main result in [6]. The difference lies in the definition of the feedback $k(\cdot)$ from (5); the latter looks like the feedback (4), but is not the same: in the new feedback (5), β is a positive parameter and not a solution of a differential equation. This fact, as well as the absence of an auxiliary differential equation, simplify the proof of Theorem 1 here, especially the definition of the Lyapunov-type function $V(\zeta)$. The iterative numerical model-based extremum seeking algorithm is applied to stabilize the system towards the equilibrium point ζ_{β_*} where the maximum methane flow rate $Q_{\max}$ is achieved. The numerical simulations illustrate the robustness of the feedback under model uncertainties.

REFERENCES

[1] Alcaraz-González V., J. Harmand, A. Rapaport, J.-P. Steyer, V. González-Alvarez, C. Pelayo-Ortiz. Water Research, **36**, 2002, 2515–2524.

[2] Antonelli R., J. Harmand, J.-P. Steyer, A. Astolfi. IEEE Trans. Control Systems Tech., **11**, 2003, No 4, 495–504.

[3] Arsie A., Ch. Ebenbauer. Journal of Differential Equations, **248**, 2010, No 10, 2458–2469.

[4] Ariyur K. B., M. Krstić. Real-time Optimization by Extremum-seeking Control. New Jersey, John Wiley & Sons, Hoboken, 2003.

[5] Dimitrova N. S., M. I. Krastanov. In: A. Rauh, E. Auer (eds), Modeling, Design, and Simulation of Systems with Uncertainties (Mathematical Engineering), Chapter 13, 2011, 275–292.

[6] Dimitrova N., M. Krastanov. Int. J. Robust Nonlinear Control, **22**, 2012, No 15, 1743–1752.

[7] Grognard F., O. Bernard. Water Sci. Techn., 53, 2006, 149–157.

[8] Hess J., O. Bernard. Journal of Process Control, **18**, 2008, No 1, 71–79.

[9] Khalil H. Nonlinear Systems. New York, Macmillan Publ. Company, 1992.

[10] Maillert L., O. Bernard, J.-P. Steyer. Automatica, **40**, 2004, 1379–1385.

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