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A FORMULA FOR THE n-TH PRIME NUMBER

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Abstract

A new formula for the n-th prime number is introduced. It is based on an arithmetic function, similar to the operation "differentiation", but defined only over natural numbers.
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1. Introduction. Probably, the first explicit formula giving the n-th prime number was introduced in 1962 by L. VESHENEVSKIY in [1].

In PAULO RIBENBOIM's book [2], three other formulas for the n-th prime number are discussed, as they were introduced in 1964 by C. P. WILLANS [3], in 1971 by J. M. GANDHI [4] and also by J. Minač in an unpublished paper.

In [5,6], the author introduced four other explicit formulas, giving the n-th prime number.

Here, we will extend the list of these formulas with a new one.

Initially, let us define function sg by

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

where x is a real number.

Below, we use the fact that every natural number n has a canonical representation in the form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $p_1, \ldots, p_k$ are different prime numbers, $\alpha_1, \ldots, \alpha_k \geq 1$ are natural numbers, and $k \geq 1$.

2. Short remarks on an arithmetic function. In 1987, the author introduced an arithmetic function with properties similar to the operation "differentiation" [7,8]. Here, short remarks on this function are given and some of its properties are studied.

Let n have the above mentioned canonical form. Following [7,8], we define the function

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k} \tag{1}$$

It is similar to the operation "partial differentiation" with respect to the variable p_i, but for natural numbers only. Obviously, if p is a prime number, then from the definition it follows that

$$\delta(p) = 1. \tag{2}$$

From (1), it follows that for every natural number n

$$\delta(n) = \sum_{i=1}^{k} \alpha_i \frac{n}{p_i}.$$

Hence,

$$\delta(n) = n. \sum_{i=1}^{k} \frac{\alpha_i}{p_i}.$$

Theorem 1 ([7,8])**.** *For every two natural numbers m and n, it holds that*

(*a*) $\delta(m.n) = \delta(m).n + m.\delta(n)$;

(*b*) *if* $\frac{m}{n}$ *is a natural number, then*

$$\delta\left(\frac{m}{n}\right) = \frac{\delta(m).n - m.\delta(n)}{n^2};$$

(*c*) $\delta(m^n) = n.m^{n-1}.\delta(m)$.

Now, it can be directly seen that for every natural number n

$$1 \leq \delta(n) < \infty$$

and for a non-prime number n

$$\delta(n) > 1.$$

For example,

$$\delta(p.q) = p + q \geq 5,$$
$$\delta(p^2) = 2p \geq 4,$$

where p and q are different prime numbers.

3. A new formula for p_n. It is well-known that function π determines the number of the prime numbers that are less than or equal to n (see, e.g., [2,9]) where $\pi(0) = 0$ and $\pi(1) = 0$.

Now, we use the definition of δ for constructing a new formula for the n-th prime number p_n.

Theorem 2. *The following equality holds for every natural number $n \geq 2$:*

$$\pi(n) = \sum_{k=2}^{n} \left[\frac{1}{\delta(k)} \right], \tag{3}$$

where $[x]$ is the integer part of the real number x.

Proof. Let $k \leq n$ be a natural number. If k is prime, then from (2)

$$\left[\frac{1}{\delta(k)} \right] = 1.$$

On the other hand, if k is not prime, then $\delta(k) > 1$, i.e.

$$\left[\frac{1}{\delta(k)} \right] = 0.$$

Therefore, the sum in the right-hand side of (3) is equal to $\pi(n)$.

Theorem 3. *For every natural number n*

$$p_n = \sum_{i=0}^{C(n)} sg(n - \pi(i)), \tag{4}$$

where (see [10], page 90, 5.27)

$$C(n) = \left[\frac{n^2 + 3n + 4}{4} \right].$$

Proof. Let us remind of the fact (see [10]) that

$$p_n < C(n)$$

for every natural number n. Now, we can see that for a fixed natural nuumber n, the function $n - \pi(i)$ is monotonically decreasing in respect of i, and so is valid for $sg(n-\pi(i))$. When $i=0,1,\ldots,p_n-1$, the values of $sg(n-\pi(i))$ are equal to 1, and when $i=p_n$, the expression $sg(n-\pi(i))$ is equal to 0.

Therefore, the sum in the right-hand side of (4) contains exactly p_n units.

Finally, from (3) and (4) we obtain

$$p_n = \sum_{i=0}^{C(n)} sg\left(n - \sum_{k=2}^{i}\left[\frac{1}{\delta(k)}\right]\right).$$

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