A FORMULA FOR THE n-TH PRIME NUMBER

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Abstract

A new formula for the n-th prime number is introduced. It is based on an arithmetic function, similar to the operation “differentiation”, but defined only over natural numbers.

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1. Introduction. Probably, the first explicit formula giving the n-th prime number was introduced in 1962 by L. Veshenevski in [1].

In PAULO RIBENBOIM’s book [2], three other formulas for the n-th prime number are discussed, as they were introduced in 1964 by C. P. WILLANS [3], in 1971 by J. M. GANDHI [4] and also by J. Minač in an unpublished paper.

In [5,6], the author introduced four other explicit formulas, giving the n-th prime number.

Here, we will extend the list of these formulas with a new one.

Initially, let us define function \( sg \) by

\[
sg(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
1, & \text{if } x > 0 
\end{cases}
\]

where \( x \) is a real number.
Below, we use the fact that every natural number \( n \) has a canonical representation in the form
\[
n = \prod_{i=1}^{k} p_i^{\alpha_i},
\]
where \( p_1, \ldots, p_k \) are different prime numbers, \( \alpha_1, \ldots, \alpha_k \geq 1 \) are natural numbers, and \( k \geq 1 \).

2. Short remarks on an arithmetic function. In 1987, the author introduced an arithmetic function with properties similar to the operation “differentiation” \([7, 8]\). Here, short remarks on this function are given and some of its properties are studied.

Let \( n \) have the above mentioned canonical form. Following \([7, 8]\), we define the function
\[
\delta(n) = \sum_{i=1}^{k} \alpha_i p_i^{\alpha_i} p_i^{\alpha_i - 1} p_i^{\alpha_i - 2} \ldots p_i^{\alpha_1 + 1} \ldots p_k^{\alpha_k}.
\]

It is similar to the operation “partial differentiation” with respect to the variable \( p_i \), but for natural numbers only. Obviously, if \( p \) is a prime number, then from the definition it follows that
\[
\delta(p) = 1.
\]

From (1), it follows that for every natural number \( n \)
\[
\delta(n) = \sum_{i=1}^{k} \alpha_i \frac{n}{p_i}.
\]

Hence,
\[
\delta(n) = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}.
\]

**Theorem 1 ([7, 8]).** For every two natural numbers \( m \) and \( n \), it holds that
(a) \( \delta(m.n) = \delta(m).n + m.\delta(n) \);
(b) if \( \frac{m}{n} \) is a natural number, then
\[
\delta\left(\frac{m}{n}\right) = \frac{\delta(m).n - m.\delta(n)}{n^2};
\]
(c) \( \delta(m^n) = n.m^{n-1}.\delta(m) \).

Now, it can be directly seen that for every natural number \( n \)
\[
1 \leq \delta(n) < \infty
\]
and for a non-prime number \( n \)
\[ \delta(n) > 1. \]

For example,
\[ \delta(p.q) = p + q \geq 5, \]
\[ \delta(p^2) = 2p \geq 4, \]
where \( p \) and \( q \) are different prime numbers.

3. A new formula for \( p_n \). It is well-known that function \( \pi \) determines the number of the prime numbers that are less than or equal to \( n \) (see, e.g., \([2,9]\)) where \( \pi(0) = 0 \) and \( \pi(1) = 0 \).

Now, we use the definition of \( \delta \) for constructing a new formula for the \( n \)-th prime number \( p_n \).

**Theorem 2.** The following equality holds for every natural number \( n \geq 2 \):

\begin{equation}
\pi(n) = \sum_{k=2}^{n} \left\lfloor \frac{1}{\delta(k)} \right\rfloor,
\end{equation}

where \( \lfloor x \rfloor \) is the integer part of the real number \( x \).

**Proof.** Let \( k \leq n \) be a natural number. If \( k \) is prime, then from (2)
\[ \left\lfloor \frac{1}{\delta(k)} \right\rfloor = 1. \]

On the other hand, if \( k \) is not prime, then \( \delta(k) > 1 \), i.e.
\[ \left\lfloor \frac{1}{\delta(k)} \right\rfloor = 0. \]

Therefore, the sum in the right-hand side of (3) is equal to \( \pi(n) \).

**Theorem 3.** For every natural number \( n \)

\begin{equation}
p_n = \sum_{i=0}^{\frac{C(n)}{4}} \text{sg}(n - \pi(i)),
\end{equation}

where (see \([10]\), page 90, 5.27)
\[ C(n) = \left\lfloor \frac{n^2 + 3n + 4}{4} \right\rfloor. \]

**Proof.** Let us remind of the fact (see \([10]\)) that
\[ p_n < C(n) \]
for every natural number \( n \). Now, we can see that for a fixed natural number \( n \), the function \( n - \pi(i) \) is monotonically decreasing in respect of \( i \), and so is valid.
for \(sg(n - \pi(i))\). When \(i = 0, 1, \ldots, p_n - 1\), the values of \(sg(n - \pi(i))\) are equal to 1, and when \(i = p_n\), the expression \(sg(n - \pi(i))\) is equal to 0.

Therefore, the sum in the right-hand side of (4) contains exactly \(p_n\) units.

Finally, from (3) and (4) we obtain

\[
p_n = \sum_{i=0}^{C(n)} sg \left( n - \sum_{k=2}^{i} \left\lfloor \frac{1}{\delta(k)} \right\rfloor \right).
\]

REFERENCES


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