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Algèbre

IDEMPOTENT ELEMENTS OF THE ENDOMORPHISM SEMIRING OF A FINITE CHAIN

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Abstract

Idempotents yield much insight in the structure of finite semigroups and semirings. In this paper, we obtain results on (multiplicative) idempotents of the endomorphism semiring of a finite chain. We prove that the set of all idempotents with given fixed points is a semiring and we find its order. We further show that this semiring is an ideal in a well-known semiring. The construction of an equivalence relation such that any equivalence class contains just one idempotent is proposed. In our main result we prove that such equivalence class is a semiring and finds its order. We prove that the set of all idempotents with certain jump points is a semiring.

Key words: endomorphism semiring, idempotents, fixed points

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1. Introduction. It is well-known that in a finite semigroup some power of each element is an idempotent, so the idempotents are very close to a generating system of the semiring (or the semigroup). For deep results using idempotents in the representation theory of finite semigroups we refer to [1,2].

Let us briefly survey the contents of our paper. After the preliminaries, in Section 3 we show some facts about fixed points of idempotent endomorphisms of a finite chain $(\{0, 1, \ldots, n-1\}, \vee)$. The central result here is Theorem 3.5 where we prove that the set of all idempotents with exactly s fixed points $k_1, < \cdots < k_s$, $1 \leq s \leq n-1$, is a semiring of order $\displaystyle\prod_{m=1}^{s-1}(k_{m+1} - k_m)$. Moreover, this semiring is an ideal of the semiring of all endomorphisms having at least $k_1, \ldots, k_s$ as fixed points. In the next Section we consider an equivalence relation on an arbitrary finite semigroup S defined by the property that $x \sim y$ for $x, y \in S$ if and only if $x^k = y^m = e$ for some $k, m \in \mathbf{N}$ and some idempotent e of S. Then we consider the equivalence classes of the semigroup $S = \left(\widehat{\mathcal{E}}_{\mathcal{C}_n}, \cdot\right)$ which is a subsemigroup of $\mathcal{PT}_n$, see [3]. Here we investigate the so-called jump points of the endomorphism and prove that between any two fixed points k_i and k_{i+1} of an endomorphism there is a unique jump point.

The main result of the paper is Theorem 4.9 where we prove that such equivalence class is a semiring of order $C_{k_{1,1}} \left(\prod_{i=1}^{\ell-1} C_{t_i} C_{s_i}\right) C_{n-1-k_{\ell, m_\ell}}$, where C_p is the p-th Catalan number. In the last Section of the paper we consider idempotent endomorphisms with arbitrary fixed points but with given jump points. Here we prove that the set of idempotent endomorphisms with the same jump points is a semiring.

2. Preliminaries. Some basic definitions and facts concerning finite semigroups can be found in any of [1,2,4,5]. As the terminology for semirings is not completely standardized, we say what our conventions are.

An algebra $(R, +, \cdot)$ with binary operations $+$ and $\cdot$ is called *semiring* if:

- $(R, +)$ is a commutative semigroup,
- $(R, \cdot)$ is a semigroup,
- both distributive laws hold: $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for any $x, y, z \in R$.

The existence of any neutral element is not assumed (see [6]).

For a semilattice $\mathcal{M}$ the set $\mathcal{E}_{\mathcal{M}}$ of the endomorphisms of $\mathcal{M}$ is a semiring (see [7]) with respect to the addition and multiplication defined by:

- $h = f + g$ when $h(x) = f(x) \vee g(x)$ for all $x \in \mathcal{M}$,
- $h = f \cdot g$ when $h(x) = f\left(g(x)\right)$ for all $x \in \mathcal{M}$.

This semiring is called the *endomorphism semiring* of $\mathcal{M}$. In this paper all semilattices are finite chains. Following [8,9] we fix a finite chain $\mathcal{C}_n = (\{0, 1, \ldots, n-1\}, \vee)$ and denote its endomorphism semiring with $\widehat{\mathcal{E}}_{\mathcal{C}_n}$. We do not assume that $\alpha(0) = 0$ for arbitrary $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$. So, there is no zero in the endomorphism semiring $\widehat{\mathcal{E}}_{\mathcal{C}_n}$. The subsemiring $\mathcal{E}_{\mathcal{C}_n} = \mathcal{E}^0_{\mathcal{C}_n}$ of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ consisting of all endomorphisms α with property $\alpha(0) = 0$ has a zero and is studied in [7–9].

If $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ is such that $\alpha(k) = i_k$ for any $k \in \mathcal{C}_n$, then we denote α as an ordered n-tuple $\wr\, i_0, i_1, i_2, \ldots, i_{n-1} \,\wr$. Note that mappings will be composed accordingly, although we shall usually give preference to writing mappings on the right, so that $\alpha \cdot \beta$ means "first α, then β." The identity $\mathbf{i} = \wr\, 0, 1, \ldots, n-1 \,\wr$ and all constant endomorphisms $\kappa_i = \wr\, i, \dots, i \,\wr$ are obviously (multiplicative) idempotents. The element $a \in \mathcal{C}_n$ satisfying $\alpha(a) = a$ is usually called a fixed point of the endomorphism α. For other properties of the endomorphism semiring we refer to [3,7–9]. In the following sections we use some terms from the book [4] having in mind that in [3] we show that some subsemigroups of the partial transformation semigroup are indeed endomorphism semirings.

3. Idempotent endomorphisms and their fixed points. The set of all idempotents of the semiring $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ is not a semiring. For example, the endomorphisms $\alpha = \wr\, 0, 0, 2 \,\wr$ and $\beta = \wr\, 0, 1, 1 \,\wr$ are idempotents of the semiring $\widehat{\mathcal{E}}_{\mathcal{C}_3}$ but $\alpha\beta = \wr\, 0, 0, 1 \,\wr$ is not an idempotent. The following four facts are well-known or are consequences from well-known facts.

Proposition 3.1. *The endomorphism $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ is an idempotent if and only if for any $k \in \mathcal{C}_n$, which is not a fixed point of α, the image $\alpha(k)$ is a fixed point of α.*

Note that in [10] maps α_k are considered, such that $\alpha_k(x) = k$ for all $x \in \mathcal{M}$, where $\mathcal{M}$ is not necessarily a finite semilattice. These maps are called constant endomorphisms. For any $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ with one fixed point the cardinality of $\mathrm{Im}(\alpha)$ may be any number from 2 to $n-1$. Indeed, for $\alpha = \wr\, n-2, n-1, \dots, n-1 \,\wr$ with unique fixed point $n-1$ we have $|\,\mathrm{Im}(\alpha)| = 2$ and for $\beta = \wr\, 1, 2, \dots, n-2, n-1, n-1 \,\wr$ with the same unique fixed point, $|\,\mathrm{Im}(\beta)| = n-1$. So, the following consequence is important.

Corollary 3.2. *An endomorphism with only one fixed point is an idempotent if and only if it is a constant.*

By similar reasonings it follows

Corollary 3.3. *An endomorphism $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ with s fixed points $k_1, \dots, k_s$, $1 \le s \le n-1$, is an idempotent if and only if $\mathrm{Im}(\alpha) = \{k_1, \dots, k_s\}$.*

Let $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ have just $n-1$ fixed points. Then $\alpha \neq \mathbf{i}$ and $n-1 \le |\,\mathrm{Im}(\alpha)| < n$. So, $|\,\mathrm{Im}(\alpha)| = n-1$ and from Corollary 3.3 it follows

Corollary 3.4. *Every endomorphism $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ with $n-1$ fixed points is an idempotent.*

The main result of this section is

Theorem 3.5. *The subset of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$, $n \ge 3$, of all idempotent endomorphisms with s fixed points $k_1 < \dots < k_s$, $1 \le s \le n-1$, is a semiring of order*

$$\prod_{m=1}^{s-1} (k_{m+1} - k_m).$$

The semiring of the idempotent endomorphisms of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ with s fixed points $k_1, \dots, k_s$ is denoted by $\mathcal{ID}(k_1, \dots, k_s)$. Since the semiring of all endomorphisms with fixed points $k_1, \dots, k_s$ is $\displaystyle\bigcap_{r=1}^{s} \mathcal{E}_{\mathcal{C}_n}^{(k_r)}$, it follows that $\mathcal{ID}(k_1, \dots, k_s)$ is a subsemiring of $\bigcap_{r=1}^{s} \mathcal{E}_{\mathcal{C}_n}^{(k_r)}$.

Corollary 3.6. *Let $n \geq 3$ and $k_1, \dots, k_s \in \mathcal{C}_n$, where $s = 1, \dots, n-2$. The semiring $\mathcal{ID}(k_1, \dots, k_s)$ is an ideal of $\bigcap_{r=1}^{s} \mathcal{E}_{\mathcal{C}_n}^{(k_r)}$.*

4. Roots of idempotent endomorphisms. Let $(S, \cdot)$ be a finite semigroup. It is well known, see [11], that for any $x \in S$ there is a positive integer $k = k(x)$ such that a^k is an idempotent element of S.

Now we consider the following relation: For any $x, y \in S$ define

$$x \sim y \iff \exists\, k, m \in \mathbf{N},\ x^k = y^m = e,$$

where e is an idempotent element of S.

Obviously the relation $\sim$ is reflexive and symmetric.

Let $x \sim y$ and $y \sim z$. Then $x^k = y^m = e_1$ and $y^r = z^s = e_2$, where e_1 and e_2 are idempotents. Now it follows $(y^m)^r = e_1^r = e_1$ and $(y^r)^m = e_2^m = e_2$, thus $e_1 = e_2$. Hence $x^k = z^s = e_1$, that is $x \sim z$. So, we prove that $\sim$ is an equivalence relation on S.

Note that two different idempotents belong to different equivalence classes modulo $\sim$. If e is an idempotent, the elements of the equivalence class containing e are called *roots of the idempotent e*. The following natural question arises:

Are there any finite noncommutative semigroups such that the equivalence classes (*modulo the above relation*) *are semigroups*?

Of course we must avoid trivial examples as groups or nilpotent semigroups.

To answer the question we consider the semigroup $\left(\widehat{\mathcal{E}}_{\mathcal{C}_n}, \cdot\right)$ and the defined above equivalence relation.

Let $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$. The element $\mathbf{j} \in \mathcal{C}_n$ is called a jump point of α if $\mathbf{j} \neq 0$ and one of the following conditions hold:

1. $\alpha(\mathbf{j} - 1) \leq \mathbf{j} - 1$ and $\alpha(\mathbf{j}) > \mathbf{j}$,
2. $\alpha(\mathbf{j} - 1) < \mathbf{j} - 1$ and $\alpha(\mathbf{j}) \geq \mathbf{j}$.

There are endomorphisms without jump points, e.g., the identity $\mathbf{i}$ and the constant endomorphisms $\kappa_k = \wr\, k\, k\, \dots\, k\, \wr$, $k = 0, 1, \dots, n-1$. For $k, \ell \in \mathcal{C}_n$, $k < \ell$, the endomorphism $\alpha_j(i) = \begin{cases} k, & i \leq j-1 \\ \ell, & i \geq j \end{cases}$ has no jump points if $j > \ell$.

Theorem 4.1. *Let $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$, $n \geq 3$, be an endomorphism with s fixed points $k_1 < \dots < k_s$, $1 \leq s \leq n-2$. Let for some i, $i = 1, \dots, s-1$, the fixed points k_i and k_{i+1} be not consecutive, i.e., $k_{i+1} \neq k_i + 1$. Then there is a unique jump point $\mathbf{j}_i$ of α such that $k_i + 1 \leq \mathbf{j}_i \leq k_{i+1}$.*

Corollary 4.2. *Every endomorphism $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$, $n \geq 3$, with just ℓ fixed points $k_1 < \cdots < k_\ell$, which are not consecutive, i.e. $k_{i+1} \neq k_i + 1$ for $i = 1, \ldots, \ell - 1$, has just $\ell - 1$ jump points $\mathbf{j}_i$ such that $k_i + 1 \leq \mathbf{j}_i \leq k_{i+1}$.*

To describe precisely all the fixed points of an arbitrary endomorphism $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ we shall use new indices. Let the first fixed point of α be $k_{1,1}$ and some fixed points after $k_{1,1}$ be consecutive, i.e., $k_{1,2}, \ldots, k_{1,m_1}$ are fixed points such that $k_{1,i+1} - k_{1,i} = 1$ for $i = 1, \ldots, m_1 - 1$. Let the next fixed point $k_{2,1}$ be such that $k_{2,1} - k_{1,m_1} > 1$. So we construct the first pair of two fixed points which are not consecutive. Let the following fixed points be $k_{2,2}, \ldots, k_{2,m_2}$ such that $k_{2,i+1} - k_{2,i} = 1$ where $i = 1, \ldots, m_2 - 1$. The next fixed point is $k_{3,1}$ and $k_{3,1} - k_{2,m_2} > 1$. Let the last pair of two not consecutive fixed points be $k_{\ell-1,m_{\ell-1}}$ and $k_{\ell,1}$. Then the last fixed points are $k_{\ell,2}, \ldots, k_{\ell,m_\ell}$ such that $k_{\ell,i+1} - k_{\ell,i} = 1$, where $i = 1, \ldots, m_\ell - 1$. So, we construct a partition of a set of fixed points of α such that we may distinguish the fixed points which are not consecutive.

Let $\mathbf{j}_{i,t_i}$ be the jump points of α such that $\mathbf{j}_{i,t_i} = k_{i,m_i} + t_i$, where $t_i = 1, \ldots, k_{i+1,1} - k_{i,m_i}$ and $i = 1, \ldots, \ell - 1$.

An endomorphism α with fixed points $k_{1,1}, \ldots, k_{\ell,m_\ell}$ and jump points $\mathbf{j}_{i,t_i}$ from the previous definitions is called endomorphism of type

$$(1) \qquad [k_{1,1}, \ldots, k_{1,m_1}, \mathbf{j}_{1,t_1}, k_{2,1}, \ldots, k_{\ell-1,m_{\ell-1}}, \mathbf{j}_{\ell-1,t_{\ell-1}}, k_{\ell,1}, \ldots, k_{\ell,m_\ell}].$$

Let us consider the endomorphism ε of this type such that:

- $\varepsilon(x) = k_{1,1}$ for any $0 \leq x \leq k_{1,1}$;
- $\varepsilon(x) = k_{i,m_i}$ for any $k_{i,m_i} \leq x \leq \mathbf{j}_{i,t_i} - 1$, where $i = 1, \ldots, \ell - 1$;
- $\varepsilon(x) = k_{i+1,1}$ for any $\mathbf{j}_{i,t_i} \leq x \leq k_{i+1,1}$, where $i = 1, \ldots, \ell - 1$;
- $\varepsilon(x) = k_{\ell,m_\ell}$ for any $k_{\ell,m_\ell} \leq x \leq n - 1$.

Now it is easy to show that this endomorphism is an idempotent.

Let $\bar{\varepsilon}$ be another idempotent of the same type. Then using the reasonings just before Corollary 5.2 it follows that:

- $\bar{\varepsilon}(x) = k_{1,1}$ for any $0 \leq x \leq k_{1,1}$;
- $\bar{\varepsilon}(x) = k_{\ell,m_\ell}$ for any $k_{\ell,m_\ell} \leq x \leq n - 1$.

Since $\bar{\varepsilon}$ is an idempotent we conclude that for some x, where $k_{i,m_i} \leq x \leq k_{i+1,1}$, it follows either $\bar{\varepsilon}(x) = k_{i,m_i}$, or $\bar{\varepsilon}(x) = k_{i+1,1}$. But using that $\bar{\varepsilon}$ is of type (1), it follows that $\bar{\varepsilon}(x) = \varepsilon(x)$ for all x, where $k_{i,m_i} \leq x \leq k_{i+1,1}$ and $i = 1, \ldots, \ell - 1$.

Hence, there is only one idempotent of a considered type (1). This endomorphism is

$$
\varepsilon = \wr\, k_{1,1}, \ldots, \underset{\underset{k_{1,m_1}}{\uparrow}}{k_{1,m_1}}, \ldots, k_{1,m_1}, \underset{\underset{\mathbf{j}_{1,t_1}}{\uparrow}}{k_{2,1}}, \ldots, k_{\ell,1}, \ldots, \underset{\underset{\mathbf{j}_{\ell-1,t_{\ell-1}}}{\uparrow}}{k_{\ell,m_\ell}}, \ldots, \underset{\underset{k_{\ell,m_\ell}}{\uparrow}}{k_{\ell,m_\ell}} \,\wr\,. \tag{2}
$$

Lemma 4.3. *Let $\varepsilon \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ be an idempotent endomorphism. If $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ is a root of ε, then the endomorphisms α and ε have the same fixed points.*

Lemma 4.4. *Let $\varepsilon \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ is an idempotent endomorphism. If $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ is a root of ε, then the endomorphisms α and ε have the same jump points.*

Immediately from Lemmas 4.3 and 4.4 it follows:

Proposition 4.5. *All endomorphisms of one equivalence class modulo $\sim$ have the same type.*

Let the idempotent endomorphism ε from (2) be an element of equivalence class E. Then E is called *equivalence class of type* (1).

Lemma 4.6. *Let E be an equivalence class of type* (1)*. Then any $\alpha \in E$ satisfies the following conditions*:

A. *$\alpha(x) > x$, where $0 \leq x < k_{1,1}$, or $\mathbf{j}_{1,t_1} \leq x < k_{2,1}$, or $\mathbf{j}_{2,t_2} \leq x < k_{3,1}$, ldots, or $\mathbf{j}_{\ell-1,t_{\ell-1}} \leq x < k_{\ell,1}$.*

B. *$\alpha(x) < x$, where $k_{1,m_1} < x < \mathbf{j}_{1,t_1}$, or $k_{2,m_2} < x < \mathbf{j}_{2,t_2}$, $\ldots$, or $k_{\ell-1,m_{\ell-1}} < x < \mathbf{j}_{\ell-1,t_{\ell-1}}$, or $k_{\ell,m_\ell} < x \leq n-1$.*

Let us remind that the jump points, which we use in (1) are defined by equality $\mathbf{j}_{i,t_i} = k_{i,m_i} + t_i$. So it follows $\mathbf{j}_{i,t_i} - k_{i,m_i} = t_i$. Now we denote $k_{i+1,1} - \mathbf{j}_{i,t_i} = s_i$. The indices t_i and s_i will be used in the next theorem.

Lemma 4.7. *The number of the ordered p-tuples $(i_0, \ldots, i_{p-1})$, where*

1. *$i_r \in \{0, \ldots, p-1\}$ for $r = 0, \ldots, p-1$;*

2. *$i_r \leq i_{r+1}$ for $r = 0, \ldots, p-1$;*

3. *$i_r < r$ for $r = 1, \ldots, p-1$*

is the p-th Catalan number $C_p = \dfrac{1}{p}\dbinom{2p-2}{p-1}$.

The proof of this lemma is a part of the proof of Proposition 3.4 in [9]. This result (using the so-called Dyck paths) and many other applications of Catalan numbers can be found in [8].

Lemma 4.8. *The number of the ordered p-tuples $(i_0, \ldots, i_{p-1})$, where*

1. *$i_r \in \{1, \ldots, p\}$ for $r = 0, \ldots, p-1$, $i_p = p$;*
2. *$i_r \leq i_{r+1}$ for $r = 0, \ldots, p-1$;*
3. *$i_r > r$ for $r = 0, \ldots, p-1$*

is the p-th Catalan number $C_p = \dfrac{1}{p}\dbinom{2p-1}{p-1}$.

Our main result is

Theorem 4.9. *Every equivalence class modulo* $\sim$ *of type* (1) *is a subsemiring of* $\widehat{\mathcal{E}}_{\mathcal{C}_n}$, $n \geq 2$. *The order of this semiring is*

$$C_{k_{1,1}} \left(\prod_{i=1}^{\ell-1} C_{t_i} C_{s_i} \right) C_{n-1-k_{\ell,m_\ell}},$$

where C_p *is the* p*-th Catalan number.*

5. The crucial role of the jump points. Here we consider idempotent endomorphisms with arbitrary fixed points but assume that each endomorphism have $\mathbf{j}_1, \dots, \mathbf{j}_r$ for jump points.

Two jump points $\mathbf{j}_s$ and $\mathbf{j}_{s+1}$ of the idempotent ε are called consecutive if $\mathbf{j}_{s+1} = \mathbf{j}_s + 1$. First let us answer the question:

Are there consecutive jump points of the idempotent endomorphism?

Yes, for instance $\varepsilon = \wr\, 1, 1, 1, 3, 5, 5 \,\wr \in \widehat{\mathcal{E}}_{\mathcal{C}_6}$ is an idempotent and the jump points 3 and 4 are consecutive. Note that 3 is also a fixed point of ε. So, we modify the question:

If the first jump point is not a fixed point, is it possible the next point to be a jump point?

The answer is negative. Indeed, if ε is an idempotent and $\varepsilon(\mathbf{j}) = k > j$, then from Proposition 3.1 it follows $\varepsilon(k) = k$ and since $\mathbf{j} + 1 \leq k$ we have $\varepsilon(\mathbf{j} + 1) \leq k$. So, $\mathbf{j} + 1$ is not a jump point of ε.

To prove the main result of this section we need three more lemmas.

Lemma 5.1. *Let* ε *be an idempotent and* $\mathbf{j}_s$ *and* $\mathbf{j}_{s+1}$ *be nonconsecutive jump points of* ε. *Then in the interval* $[\mathbf{j}_s, \mathbf{j}_{s+1} - 1]$ *one of the following holds*:

1. ε *is a constant endomorphism.*
2. ε *is an identity.*
3. ε *is an identity in the interval* $[\mathbf{j}_s, k]$ *and a constant endomorphism in the interval* $[k, \mathbf{j}_{s+1} - 1]$.
4. ε *is a constant endomorphism in the interval* $[\mathbf{j}_s, k]$ *and an identity in the interval* $[k, \mathbf{j}_{s+1} - 1]$.
5. ε *is a constant endomorphism in the interval* $[\mathbf{j}_s, k]$, *an identity in the interval* $[k, \ell]$ *and a constant endomorphism in the interval* $[\ell, \mathbf{j}_{s+1} - 1]$.

It is straightforward to show that if one of the conditions 1–5 of Lemma 5.1 holds for an endomorphism ε, then ε is an idempotent.

Lemma 5.2. *Let* ε *and* $\bar{\varepsilon}$ *be idempotent endomorphisms of* $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ *with the same jump points* $\mathbf{j}_1, \dots, \mathbf{j}_r$. *Then* $\varepsilon + \bar{\varepsilon}$ *is also an idempotent endomorphism with the same jump points.*

Lemma 5.3. *Let ε and $\bar{\varepsilon}$ be idempotent endomorphisms of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ with the same jump points $\mathbf{j}_1, \dots, \mathbf{j}_r$. Then $\varepsilon \cdot \bar{\varepsilon}$ is also an idempotent endomorphism with the same jump points.*

Immediately from Lemma 5.2 and Lemma 5.3 it follows:

Theorem 5.4. *The set of idempotent endomorphisms of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ with the same jump points is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.*

Using Lemma 5.2 and Lemma 5.3 it is easy to prove

Proposition 5.5. *The set of idempotent endomorphisms of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ without jump points is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ of order $\binom{n+1}{2}$.*

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