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NON-ITERATIVE IMPROVEMENT OF TRACE BOUNDS FOR THE LYAPUNOV EQUATION SOLUTION

Svetoslav Savov, Ivan Popchev

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Abstract

The improvement problem for available lower and upper trace bounds for the solution of the continuous-time Lyapunov equation (LE) is investigated. It is shown how an arbitrary positive (semi)-definite lower matrix solution bound can be used to get always tighter trace estimates.

Key words: Lyapunov equation, solution trace bounds

1. Introduction. The estimation problem for the solution of the LE attracts the attention for more than half a century. The current interest is motivated both by theoretical and practical reasons. Upper and lower, scalar and matrix bounds provide some essential information about the dependence of the solution size on the equation parameters, i.e., the coefficient matrix and the right-hand side positive (semi)-definite matrix. The LE plays an essential role in contemporary control theory. Sometimes, it is either impossible, or even not necessary to compute the exact solution [3]. In these cases, it is desirable to get some bounds for it. The solution estimation problem has two main aspects – bounds validity restrictions and, of coarse, bounds tightness.

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This paper investigates the possibility to improve available in the literature valid trace bounds for the solution of LE. The main result consists in the proposed new lower and upper bounds.

2. Preliminaries. The following notations will be used: if X and Y are symmetric matrices, $X \geq Y$ means that $X - Y$ is a positive semi-definite matrix; $\lambda_1(X)$, $\lambda_n(X)$ denote the maximal and the minimal eigenvalue of an $n \times n$ symmetric matrix X, respectively; for a square matrix X, $X_s = 0.5(X^T + X)$ and I is the identity matrix of a respective dimension. The inverse of a nonsingular matrix and the square root of a positive (semi)-definite matrix are denoted by X^{-1}, and $X^{1/2}$, respectively. Recall also some of the basic properties of the trace (sum of eigenvalues = sum of diagonal entries) operator. For arbitrary square matrices X and Y and a symmetric matrix Z of the same dimension, one has

$$\mathrm{tr}(X) = \mathrm{tr}(X^T), \quad \mathrm{tr}(X+Y) = \mathrm{tr}(X) + \mathrm{tr}(Y), \quad \mathrm{tr}(XY) = \mathrm{tr}(YX),$$
$$\mathrm{tr}(XZ) = \mathrm{tr}(X_s Z).$$

3. The trace bounds problem. It is well known that a matrix A is a positive stable one if and only if the LE

$$A^T P + PA = -2Q \tag{1}$$

has a unique positive (semi)-definite solution matrix P for any given positive (semi)-definite matrix Q. The improvement of some well known trace bounds for the solution of (1) is studied in this research. While various, always valid lower bounds can be easily derived, the existing upper bounds validity depends on the satisfaction of some rather restrictive assumptions concerning the coefficient matrix A. Consider the following trace bounds [7]:

$$\frac{\mathrm{tr}(Q)}{-\lambda_n(A_s)} = b_{L_0} \leq \mathrm{tr}(P) \leq b_{U_0} = \frac{\mathrm{tr}(Q)}{-\lambda_1(A_s)}. \tag{2}$$

Validity of the lower and upper bound is guaranteed if $\lambda_n(A_s) < 0$, $\lambda_1(A_s) < 0$, respectively. Since A is a Hurwitz (negative stable) matrix, then $\mathrm{tr}(A) = \mathrm{tr}(A_s) < 0$, which means that $\lambda_n(A_s) < 0$ by necessity. The upper bound's validity condition is rather restrictive one and does not necessarily hold in the general case. Till 2004, all reported upper bounds were valid under the strong restriction that $A_s < 0$ [1,2]. By making use of the singular value decomposition (s.v.d.) of the coefficient matrix $A = U\Sigma V^T$, $UU = VV^T = I$, where Σ is a diagonal matrix containing the singular values of A, various upper and lower bounds have been suggested in [4,5], provided that $F = UV^T$ is a Hurwitz matrix, which is equivalent to $F_s < 0$, since F is unitary. It was also proved that $A_s < 0$

and always implies $F_s < 0$. This simple fact helps the extension of the set of Hurwitz matrices

(3)
$$A = R_1F = FR_2,\ R_1 = U\Sigma U^T = (AA^T)^{1/2},\ R_2 = (A^TA)^{1/2},\ F = UV^T,\ F_s < 0$$

for which valid bounds for the solution of (1) can be obtained. The next lower and upper trace bounds have been proved in [6]

$$\max(t_{L_1}, t_{L_2}) = b_{L_1} \le \mathrm{tr}(P) \le b_{U_1} = \min(t_{U_1}, t_{U_2}), \tag{4}$$
$$t_{L_1} = \frac{\mathrm{tr}(QR_1)}{-\lambda_n(R_1F_sR_1)};\ t_{L_2} = \frac{\mathrm{tr}(QR_2^{-1})}{-\lambda_n(F_s)};\ t_{U_1} = \frac{\mathrm{tr}(QR_1)}{-\lambda_1(R_1F_sR_1)};\ t_{U_2} = \frac{\mathrm{tr}(QR_2^{-1})}{-\lambda_1(F_s)}.$$

The lower bound in (4) is always valid since $\mathrm{tr}(A) = \mathrm{tr}(A_s) = \mathrm{tr}(F_sR_1) = \mathrm{tr}(F_sR_2) < 0$ and therefore $\lambda_n(F_s) < 0 \Leftrightarrow \lambda_n(R_1F_sR_1) < 0$. The upper bound in (4) is valid if $F_s < 0$ and, as it was already said, this is a less restrictive requirement than $A_s < 0$. In other words, the upper bound in (4) may be valid even if the upper bound in (2) is not. Thus, the usage of the s.v.d. of the coefficient matrix A contributes to the extension of the set of Hurwitz matrices for which valid upper trace solution bounds exist. Unfortunately, it is not possible to compare the bounds (2) and (4), in the sense of tightness, in the general case. After these preliminary words concerning the imposed on matrix A bounds validity restrictions, the problem of their improvement will be considered next.

4. New tighter trace bounds. Suppose that there exists some positive (semi)-definite lower matrix bound for the solution P in (1). It will be shown how it can be used to improve the tightness of both the lower and upper trace bounds in (2) and (4).

Lemma 4.1. *For any matrix P_L, $0 \le P_L \le P$, the solution trace can be estimated as follows:*

$$\max(t_{L_3}, t_{L_4}, t_{L_5}) = b_{L_2} \le \mathrm{tr}(P) \le b_{U_2} = \min(t_{U_3}, t_{U_4}, t_{U_5}), \tag{5}$$
$$t_{L_3} = \frac{\mathrm{tr}[Q + (A_s - \lambda_n(A_s)I)P_L]}{-\lambda_n(A_s)};$$
$$t_{U_3} = \frac{\mathrm{tr}[Q + (A_s - \lambda_1(A_s)I)P_L]}{-\lambda_1(A_s)}\ \textit{if}\ \lambda_1(A_s) < 0;$$
$$t_{L_4} = \frac{\mathrm{tr}[QR_1 + (R_1F_sR_1 - \lambda_n(R_1F_sR_1)I)P_L]}{-\lambda_n(R_1F_sR_1)};$$
$$t_{U_4} = \frac{\mathrm{tr}[QR_1 + (R_1F_sR_1 - \lambda_1(R_1F_sR_1)I)P_L]}{-\lambda_1(R_1F_sR_1)}\ \textit{if}\ \lambda_1(F_s) < 0;$$
$$t_{L_5} = \frac{\mathrm{tr}[QR_2^{-1} + (F_s - \lambda_n(F_s)I)P_L]}{-\lambda_n(F_s)};$$

$$t_{U_4} = \frac{\mathrm{tr}[QR_2^{-1} + (F_s - \lambda_1(F_s)I)P_L]}{-\lambda_1(F_s)} \quad \textit{if } \lambda_1(F_s) < 0.$$

Proof. Consider the modified LE

$$A^T(P - P_L) + (P - P_L)A = -(2Q + A^T P_L + P_L A) = -2\tilde{Q}. \tag{6}$$

Since $P - P_L \geq 0$, by assumption, and having in mind the trace bounds (2) and (4), the trace of the positive (semi)-definite difference solution matrix in (6) can be bounded as follows:

$$\frac{\mathrm{tr}(\tilde{Q})}{-\lambda_n(A_s)} \leq \mathrm{tr}(P - P_L) \leq \frac{\mathrm{tr}(\tilde{Q})}{-\lambda_1(A_s)}; \tag{7}$$

$$\frac{\mathrm{tr}(\tilde{Q}R_1)}{-\lambda_n(R_1F_sR_1)} \leq \mathrm{tr}(P - P_L) \leq \frac{\mathrm{tr}(\tilde{Q}R_1)}{-\lambda_1(R_1F_sR_1)}; \tag{8}$$

$$\frac{\mathrm{tr}(\tilde{Q}R_2^{-1})}{-\lambda_n(F_s)} \leq \mathrm{tr}(P - P_L) \leq \frac{\mathrm{tr}(\tilde{Q}R_2^{-1})}{-\lambda_1(F_s)}. \tag{9}$$

Having in mind the inequalities in (7), the trace bounds t_{L_3}, t_{U_3} are easily obtained. Consider the s.v.d. of A in (3). The traces in (8) and (9) can be rewritten as follows:

$$\begin{aligned}
\mathrm{tr}(\tilde{Q}R_1) &= \mathrm{tr}[QR_1 + 0.5(F^T R_1 P_L + P_L R_1 F)R_1] \\
&= \mathrm{tr}[QR_1 + 0.5(R_1 F^T R_1 P_L + P_L R_1 F R_1)] \\
&= \mathrm{tr}(QR_1 + R_1 F_s R_1 P_L); \\
\mathrm{tr}(\tilde{Q}R_2^{-1}) &= \mathrm{tr}[QR_2^{-1} + 0.5(R_2 F^T P_L + P_L F R_2)R_2^{-1}] \\
&= \mathrm{tr}[QR_2^{-1} + 0.5(F^T P_L + P_L F)] \\
&= \mathrm{tr}(QR_1 + F_s P_L).
\end{aligned}$$

The bounds t_{L_4}, t_{U_4}, t_{L_5}, t_{U_5} are obtained by making use of these trace equalities and the respective inequalities in (8) and (9), which proves the lower and upper trace estimates in (5).

The next result proves that the new bounds are always tighter than the respective ones in (2) and (4).

Corollary 4.1. *For any given matrix P_L, $0 \leq P_L \leq P$, one has*

$$b_{L_2} \geq \max(b_{L_0}, b_{L_1}), \quad b_{U_2} \leq \min(b_{U_0}, b_{U_1}). \tag{10}$$

In addition, if P_L is a strictly positive definite lower matrix bound for P, then any equality in (10) *is possible if and only if* $b_{L_2} = b_{U_2} = \mathrm{tr}(P)$.

Proof. The bounds defined in (5) can be rewritten as follows:

$$t_{L3} = b_{L_0} + \delta_{L_3},\quad t_{L_4} = b_{L_1} + \delta_{L_4},\quad t_{L_5} = b_{L_2} + \delta_{L_5},\quad \delta_{L_i} \geq 0,\ i = 3,4,5;$$
$$t_{U_3} = b_{U_0} + \delta_{U_3},\quad t_{U_4} = b_{U_1} + \delta_{U_4},\quad t_{U_5} = b_{U_2} + \delta_{U_5},\quad \delta_{U_i} \leq 0,\ i = 3,4,5,$$

since for any given $n \times n$ symmetric matrix X, one has

$$X - \lambda_n(X)I \geq 0,\quad X - \lambda_1(X)I \leq 0.$$

This proves the inequalities in (10). Let $P_L > 0$ and suppose that $b_{L_2} = \max(b_{L_0}, b_{L_1})$. This is possible in some of the following three cases:

Case 1. $\delta_{L_3} = 0 \Leftrightarrow A_s = \lambda_n(A_s)I = \lambda_1(A_s)I \Rightarrow \mathrm{tr}(P) = t_{L_3} = t_{L_0} = t_{U_3} = t_{U_0}$.

Case 2. $\delta_{L_4} = 0 \Leftrightarrow 2R_1F_sR_1 = R_1F^TR_1 + R_1FR_1 = R_1A^T + AR_1 = 2\lambda_n(R_1F_sR_1)I = 2\lambda_1(R_1F_sR_1)I$.

Consider the LE (1) multiplied by R_1 from the right, i.e.

$$A^TPR_1 + PAR_1 = -2QR_1 \Rightarrow \mathrm{tr}[(R_1A^T + AR_1)P] = 2\lambda_n(R_1F_sR_1)\,\mathrm{tr}(P)$$
$$= 2\lambda_1(R_1F_sR_1)\,\mathrm{tr}(P) = -2\,\mathrm{tr}(QR_1).$$

It follows that $\mathrm{tr}(P) = t_{L_4} = t_{L_1} = t_{U_4} = t_{U_1}$, in this special case.

Case 3. $\delta_{L_5} = 0 \Leftrightarrow 2F_s = F^T + F = R_2^{-1}A^T + AR_2^{-1} = 2\lambda_n(F_s)I = 2\lambda_1(F_s)I$.

Consider the LE (1) multiplied by R_2^{-1} from the right, i.e.

$$A^TPR_2^{-1} + PAR_2^{-1} = -2QR_2^{-1} \Rightarrow \mathrm{tr}[(R_2^{-1}A^T + AR_2^{-1})P] = 2\lambda_n(F_s)\,\mathrm{tr}(P)$$
$$= 2\lambda_1(F_s)\,\mathrm{tr}(P) = -2\,\mathrm{tr}(QR_2^{-1}).$$

Therefore, $\mathrm{tr}(P) = t_{L_5} = t_{L_2} = t_{U_5} = t_{U_2}$, in this case. If $b_{U_2} = \min(b_{U_0}, b_{U_1})$, the above statement can be proved in a similar way. This completes the proof of the lemma.

REFERENCES

[1] Komaroff N. IEEE Transactions on Automatic Control, **37**, 1992, No 5, 337–341.
[2] Kwon W., Y. Moon, S. Ahn. Int. J. Control, **64**, 1996, No 3, 377–389.
[3] Mori T., H. Kokame. IEICE Trans. Fundamentals, **E-81A**, 1998, No 10, 1301–1304.
[4] Savov S., I. Popchev. Compt. rend. Acad. bulg. Sci., **59**, 2006, No 9, 933–938.

[5] Savov S., I. Popchev. Int. J. Control and Automation Systems, **6**, 2008, No 2, 288–294.
[6] Savov S., I. Popchev. Int. J. Pure Appl. Math., **49**, 2008, No 3, 381–389.
[7] Wang S., T. Kuo, C. Hsu. IEEE Transactions on Automatic Control, **31**, 1986, No 7, 654–656.

Institute of Information and Communication Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria