Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 66, No 5, 2013

MATHEMATIQUES
Analyse réelle

ABOUT A PARTIAL THETA FUNCTION

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(*Submitted by Academician P. Kenderov on December 12, 2012*)

Abstract

Partial theta functions are of interest to statistical physics and combinatorics, to Ramanujan type q-series, to asymptotic analysis and to the theory of (mock) modular forms. One such function is defined by the series $g_q(x) := \sum_{k=0}^{\infty} q^{k^2} x^k$. We study the real analytic in $(-1,1)$ function $-1+2g_q(-1)$ which appears in paper [4] when the relationship of partial theta functions and real rooted polynomials is considered.

Key words: partial theta function, hyperbolic polynomial
2000 Mathematics Subject Classification: 26A06

The series in two variables $\Psi(q,x) := \sum_{k=0}^{\infty} q^{\binom{k+1}{2}} x^k$, in which x can be considered as a variable and q as a parameter, converges for $|q| < 1$, $x \in \mathbf{C}$. It defines for each $|q| < 1$ fixed an entire function called a *partial theta function.* This terminology might refer also to $\Psi(q,-x)$ or to $\Psi(q,-x/q)$ or to $g_q(x) := \sum_{k=0}^{\infty} q^{k^2} x^k = \Psi(q^2, x/q)$.

Partial theta functions are of interest to statistical physics and combinatorics [7], to Ramanujan type q-series [8], to asymptotic analysis [2] and to the theory of (mock) modular forms [3]; see also [1]. They can be considered not only as holomorphic, but also as real analytic functions (i.e. when q and x are real).

In paper [4], when studying some properties of partial theta functions in the real analytic context and their relationship with hyperbolic polynomials (i.e. real polynomials with all roots real), the function in one variable $\psi(q) = 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2} = -1 + 2g_q(-1)$ appears. When $q \in (-1,1)$ (respectively $q \in \mathbf{C}$, $|q| < 1$), this lacunary series converges and defines a real analytic function on $(-1,1)$ (respectively a function holomorphic in the open unit disk).

It is proved in [6], Chapter 1, Problem 56, that

$$1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2} = \prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k} \ . \tag{1}$$

In the present paper we consider the real analytic situation (i.e. $q \in (-1,1)$). We prove the following theorem:

Theorem 1. *The function ψ has the properties*:

1. *One has $\psi' < 0$ for all $q \in (-1,1)$.*
2. *One has $\lim_{q\to 1}\psi(q) = 0$, $\lim_{q\to -1}\psi(q) = \infty$.*
3. *As $q \to 1$, one has $\psi(q) = o((q-1)^l)$ for any $l \in \mathbf{N}$. In other words, ψ is flat at 1.*
4. *One has $\psi'' \geq 0$ for all $q \in (-1,1)$, with equality only for $q = 0$.*
5. *The function $\tau(q) := (q-1)\ln\psi(q)$ is increasing on $(0,1)$ and $\lim_{q\to 1}\tau(q) = \pi^2/4$. More precisely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $e^{\frac{\pi^2}{4(q-1)}} < \psi(q) \leq e^{\frac{\pi^2-\varepsilon}{4(q-1)}}$ for $q \in (1-\delta, 1)$.*
6. *As $q \to -1$, one has $\psi(q) = o((q+1)^{-1})$ and $(q+1)^{\alpha}/\psi(q) = o(1)$ for any $\alpha \in (-1,0)$.*

Proof. For $q \in (-1,0)$ all terms of the series

$$\psi' = \sum_{k=1}^{\infty}(-1)^k k^2 q^{k^2-1} \ ,$$

respectively

$$\psi'' = \sum_{k=2}^{\infty}(-1)^k k^2(k^2-1)q^{k^2-2}$$

are negative (respectively positive). This proves properties 1 and 4 for $q \in (-1, 0)$. For $q = 0$ their proof follows from the values of the coefficients of q and q^2 in the series ψ (-2 and 0 respectively).

For $q \in (0, 1)$ all functions $1 - q^k$ and $1/(1 + q^k)$, $k = 1, 2, \ldots$, are positively valued and their derivatives are negatively valued (see (1)). By the Leibniz rule their product has a negatively valued derivative. This proves property 1.

For $q \in (-1, 0)$ all terms of the series $\sum_{k=1}^{\infty} (-1)^k q^{k^2}$ are positive and each of them tends to 1 as $q \to -1$. Hence $\lim_{q \to -1} \psi(q) = \infty$. Every factor $\varphi_k := (1 - x^k)/(1 + x^k)$ is a function analytic in a neighbourhood of 1 which has a simple zero at 1, therefore $\lim_{q \to 1} \psi(q) = 0$ and properties 2 and 3 hold.

Set $l(q) := \ln \psi(q)$. To settle property 4 for $q \in (0, 1)$ we use the formula $\psi'' = (e^l)'' = e^l(l'' + l'^2)$. To prove that $\psi'' > 0$ for $q \in (0, 1)$ it suffices to show that all Taylor coefficients of $l'' + l'^2$ are nonnegative. Denote by $c_j[f]$ the coefficient of q^j in the Taylor series at 0 of the function f.

Using the Taylor series $\ln(1 \pm q) = \pm q - \frac{q^2}{2} \pm \frac{q^3}{3} - \frac{q^4}{4} \pm \frac{q^5}{5} - \cdots$ one gets

$$\ln \frac{1 - q^k}{1 + q^k} = (-2) \left(q^k + \frac{q^{3k}}{3} + \frac{q^{5k}}{5} + \cdots \right) . \tag{2}$$

Hence $l(q) = -\sum_{j=0}^{\infty} d_j q^j$, where

$$2 \le d_j \le 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2[(j-1)/2] + 1} \right) < 2 \left(\sum_{i=1}^{j} \frac{1}{i} \right) \le 2(\ln j + 1) \tag{3}$$

($[\cdot]$ denotes the integer part of a real number).

One has

$$c_j[l''] = (j+2)(j+1)d_{j+2} \qquad < 2(j+2)(j+1)(\ln(j+2) + 1),$$
$$c_j[l'^2] = \sum_{k=1}^{j+1} k(j-k+2) d_k d_{j-k+2} \ge \frac{2(j+1)(j+2)(j+3)}{3}$$

(we use here the inequalities (3) and the well-known equalities $\sum_{k=1}^{j+1} k = (j+1) \times (j+2)/2$ and $\sum_{k=1}^{j+1} k^2 = (j+1)(j+2)(2j+3)/6$).

For $j \ge 8$ one has $\frac{j+3}{3} > \ln(j+2) + 1$ and, hence, $c_j[l'^2] > c_j[l'']$. For $j \le 7$ the inequality $c_j[l'^2] \ge c_j[l'']$ follows from Table 1.

Table 1

j	0	1	2	3	4	5	6	7	8	9
$c_j[-l]$	0	2	2	$2+\frac{2}{3}$	2	$2+\frac{2}{5}$	$2+\frac{2}{3}$	$2+\frac{2}{7}$	2	$2+\frac{2}{3}+\frac{2}{9}$
$c_j[-l'']$	4	16	24	48	80	96	112	208		
$c_j[-l']$	2	4	8	8	12	16	16	16		
$c_j[l'^2]$	4	16	48	96	176	288	448	640		

One has $c_0[-l''] = c_0[l^2]$ and $c_1[-l''] = c_1[l^2]$ because $\psi''(0) = \psi'''(0) = 0$.

PROVE PART 5. Summing up formula (2) for $k = 1, 2, \ldots$, one deduces the equality

$$(4) \quad l(q) = (-2)\sum_{k=1}^{\infty}\left(q^k + \frac{q^{3k}}{3} + \frac{q^{5k}}{5} + \cdots\right) = (-2)\sum_{j=0}^{\infty}\frac{q^{2j+1}}{(2j+1)(1-q^{2j+1})} .$$

Thus $\tau(q) = (q-1)l(q) = 2\sum_{j=0}^{\infty}\zeta_j(q)$, where

$$\zeta_j(q) = \frac{q^{2j+1}}{(2j+1)(1+q+q^2+\cdots+q^{2j})}.$$

It follows from $(2j+1)\zeta_j(q) = 1/(q^{-2j-1}+q^{-2j}+\cdots+q^{-1})$ that ζ_j is increasing on $(0,1)$ and tends to $1/(2j+1)^2$ as $q \to 1$. The series $\sum_{j=0}^{\infty} 1/(2j+1)^2$ is convergent and its sum is $\pi^2/8$. Hence for $q \in (0,1)$ one has $\tau(q) < \pi^2/4$, τ is increasing and $\lim_{q\to 1}\tau(q) = \pi^2/4$. This implies part 5 of the theorem.

PROVE PART 6. For $q \in (-1,0)$ every term of the series $1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2}$ is smaller than the corresponding term of the series $2\sum_{k=0}^{\infty}(-1)^k q^k = 2/(1+q)$. Hence ψ grows not faster than $2/(1+q)$ as $q \to -1$.

To see that

$$(*) \qquad \lim_{q\to -1}(1+q)\psi(q) = 0$$

write

$$(1+q)\psi(q) = 1 - q + 2\sum_{k=2}^{\infty}\kappa_k\ , \quad \text{where} \quad \kappa_k = (-1)^k(q^{k^2} + q^{k^2+1})\ .$$

The maximal value of κ_k is $\kappa_k(-k^2/(k^2+1)) = (k^2/(k^2+1))^{k^2}(1/(k^2+1)) < 1/(k^2+1)$. The series $\sum_{k=2}^{\infty} 1/(k^2+1)$ is absolutely convergent and $\lim_{q\to -1}\kappa_k(q) = 0$, which implies (*).

Consider the function ν defined by the series $q + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2+1}/(k^2+1)$. Hence $\nu' = \psi$. To avoid a sign change in the sequence of coefficients in what follows, we consider the functions $-\nu(-q)$ and $\psi(-q)$ for $q \to 1$ instead of ν and ψ for $q \to -1$.

We show that for $\alpha \in (0,1)$ one has

$$(**) \qquad \lim_{q\to 1}(-\nu(-q)/(1-q)^{\alpha}) = \infty.$$

By the L'Hôspital rule, this means that $\lim_{q\to 1}(\psi(-q)/|1-q|^{\alpha-1}) = \infty$ which is just property 6 (when α takes all values of $(0,1)$, $\alpha - 1$ takes all values of $(-1,0)$). To justify the limit (**) it suffices to observe that:

i) all nonzero coefficients of the series of $-\nu(-q)$ are positive;

ii) all coefficients of the series of $(1-q)^{-\alpha}$ are positive and their sum S is ∞.

Statement i) is evident. Statement ii) is easily deduced from

$$(1-q)^{-\alpha} = 1 + \alpha q + \frac{(-\alpha)(-\alpha-1)}{2!}q^2 + \frac{\alpha(-\alpha-1)(-\alpha-2)}{3!}q^3 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)(-\alpha-3)}{4!}q^4 + \cdots .$$

It is clear that all coefficients are positive. Set $\beta := |\alpha|$. Hence $\beta \in (0,1)$. The coefficient of q^k is

$$\beta \cdot (\beta+1) \cdot \frac{\beta+2}{2} \cdots \frac{\beta+k-1}{k-1} \cdot \frac{1}{k} > \beta(\beta+1)\frac{1}{k}$$

which shows that $S = \infty$. Hence when multiplying the series of $-\nu(-q)$ and $(1-q)^{-\alpha}$ and when letting q tend to 1, one gets as a limit the product of the positive sum of the coefficients of $-\nu(-q)$ and the "infinite positive sum S" of the coefficients of $(1-q)^{-\alpha}$, i.e. one gets ∞ as a limit.

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