Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 66, No 5, 2013

MATHEMATIQUES
Analyse fonctionnelle

ON A SWEEPING PROCESS WITH THE CONE OF LIMITING NORMALS

Bogdan Georgiev*, Nadezhda Ribarska*,**

(*Submitted by Corresponding Member J. Revalski on December 18, 2012*)

Abstract

We present two positive results on the existence of solutions of a sweeping process with the cone of limiting normals. The assumptions on the sets involve o-minimal structures.

Key words: differential inclusions, sweeping process, o-minimal structures

2010 Mathematics Subject Classification: 34A60, 49J21, 03C64

1. Introduction. The classical sweeping process has been introduced and thoroughly studied in the 70s by J. J. Moreau (cf., e.g. [1]). General motivation arising from Mechanics appeared in [2]. Two years later the first paper by Moreau, planning procedures in mathematical economy led C. Henry (see [3]) to a differential inclusion of a similar nature. Recently, Bernicot and Venel (cf. [4]) showed that modelling of crowd motion in emergency evacuation leads to the same problem. Nowadays there exists an extensive literature on the subject.

The mathematical formulation of a sweeping process is the following constrained differential inclusion:

$$(1)\qquad \begin{aligned} &\dot{x}(t) \in -N_{C(t)}(x(t)),\\ &x(0) = x_0 \in C(0),\\ &x(t) \in C(t), \end{aligned}$$

Partially supported by the St. Kliment Ohridski University Fund "Research and Development" under Contract 126/2012.

where $C(t)$ is a given moving closed set and $N_{C(t)}(x(t))$ is the normal cone (in some sense) to $C(t)$ at $x(t)$.

The investigation of this problem has been carried out under different assumptions. The phase space usually is R^n or a Hilbert space and only few works are in the general setting of an arbitrary Banach space. The set $C(t)$ is moving continuously with respect to the Hausdorff distance and in some papers additional assumptions are used as absolute continuity or Lipschitz continuity (again with respect to the Hausdorff distance). From the outset of the topic (c.f. [3]) some perturbations with "nice" functions of the right-hand side have been considered. The normal cone $N_{C(t)}(x(t))$ is assumed to be the Clarke normal cone. We refer the reader to [5] and references therein.

In finite dimension a general existence result has been independently obtained by Colombo-Goncharov (c.f. [7]) and Benabdellah (c.f. [6]):

Theorem 1.1. *The sweeping process* (1) *has a solution provided the phase space is R^n, the multifunction $C(t)$ is Lipschitz continuous with respect to the Hausdorff distance and $N_{C(t)}(x(t))$ is the Clarke normal cone.*

Let us recall the basic concepts of normal cones to a closed set $C \subset R^n$ (which are essentially the same in Hilbert spaces) at some point $x \in C$. A vector $\zeta \in R^n$ is said to be a *proximal normal* to C at $x \in C$ if there exists a positive real t such that the metric projection of the point $x + t\zeta$ on C coincides with x. The cone of all proximal normals to C at x is denoted by $\widehat{N}_C(x)$. Closing the graph of the mapping $x \mapsto \widehat{N}_C(x)$ we obtain the mapping $x \mapsto N_C(x)$ assigning to each point $x \in C$ the *cone of limiting normals* $N_C(x)$ to C at x. The cone $\overline{\text{co}}\ N_C(x)$ is said to be the *Clarke normal cone* to C at x.

We are interested in investigating the existence of solutions of (1) if the normal cone to the moving set $C(t)$ is the cone of limiting normals. This question was posed to the second named author by Jourani in 2007. It is a substantially new problem because of the nonconvexity of the right-hand side of the differential inclusion and requires new methods. Let us remark that replacing the Clarke normal cone with the cone of limiting normals makes sense in applications where feasible velocities are obtained by projecting some drift term on the tangent cone of the set (c.f. [3,4]). In this note we present two positive results in this direction (Section 3) whose main assumptions involve definable sets and mappings (preliminaries in Section 2).

2. Preliminaries. Definable and tame sets, functions and mappings are a product of model theory and algebraic geometry; they are the main concepts of the theory of the so-called "o-minimal structures" that has been actively developing during the last 20–25 years (c.f., e.g. [8,9]). Applications of this theory to optimization problems are becoming increasingly popular because the classes of sets and mappings involved are, on the one hand, broad enough to encompass essentially all the important applications and, on the other hand, small enough to avoid "pathologies" like fractals.

Definition 2.1. A structure (expanding the real closed field R) is a collection $\mathcal{S} = (\mathcal{S}^n)_{n\in N}$, where each $\mathcal{S}^n$ is a set of subsets of the space R^n, satisfying the following axioms:

1. All algebraic subsets of R^n are in $\mathcal{S}^n$. Recall that an algebraic set is a subset of R^n defined by a finite number of polynomial equations

$$P_1(x_1, \dots, x_n) = \cdots = P_k(x_1, \dots, x_n) = 0.$$

2. For every n, $\mathcal{S}^n$ is a Boolean subalgebra of the powerset 2^{R^n} of R^n.

3. If $A \in \mathcal{S}^m$ and $B \in \mathcal{S}^n$, then $A \times B \in \mathcal{S}^{m+n}$.

4. If $p : R^{n+1} \longrightarrow R^n$ is the projection on the first n coordinates and $A \in \mathcal{S}^{n+1}$, then $p(A) \in \mathcal{S}^n$.

The elements of $\mathcal{S}^n$ are called the definable subsets of R^n. The structure $\mathcal{S}$ is said to be o-minimal if, moreover, it satisfies:

5. The elements of $\mathcal{S}^1$ are precisely the finite unions of points and intervals.

In this work we always assume that the closed field R coincides with the field of the real numbers. Standard examples of o-minimal structures are the semi-algebraic sets (finite unions of sets defined by finitely many algebraic equalities and inequalities), globally subanalytic sets (this class contains all bounded sets which are finite unions of sets defined by finitely many analytic equalities and inequalities).

Any o-minimal structure enjoys magnificent stability properties, e.g. the closure and the interior of a definable subset of R^n are definable; the image of a definable set by a definable map (i.e. whose graph is a definable set) is definable. In fact, every "reasonably" defined set, that is the definition uses finite combination of quantifiers, is definable (provided quantified variables range over definable sets). The following definition and the subsequent theorem make this precise:

Definition 2.2. A first-order formula is constructed recursively according to the following rules:

1. If P is a polynomial of n variables, then

$$P(x_1, \dots, x_n) = 0 \text{ and } P(x_1, \dots, x_n) > 0$$

are first-order formulas.

2. If A is a definable subset of R^n, then $x \in A$ is a first-order formula.

3. If $\Phi(x_1, \dots, x_n)$ and $\Psi(x_1, \dots, x_n)$ are first-order formulas, then "Φ and Ψ," "Φ or Ψ," "not Φ," "$\Phi \Longrightarrow \Psi$" are first-order formulas.

4. If $\Phi(y, x)$ is a first-order formula (where $y = (y_1, \dots, y_p)$ and $x = (x_1, \dots, x_n)$) and A is a definable subset of R^n, then $\exists\ x \in A\ \Phi(y, x)$ and $\forall\ x \in A\ \Phi(y, x)$ are first-order formulas.

Theorem 2.3. *If $\Phi(x_1, \dots, x_n)$ is a first-order formula, the set of all vectors $(x_1, \dots, x_n)$ in R^n which satisfy $\Phi(x_1, \dots, x_n)$ is definable.*

The most important (from our point of view) feature of the sets belonging to some o-minimal structure is the highly nontrivial fact that they admit a regular C^k-Whitney stratification for any $k \in N$:

Definition 2.4. Let $A \subset R^n$ and $k \in N$. We say that A admits a regular C^k-Whitney stratification if there exists a finite partitioning of A into C^k manifolds $\{M_i\}_{i=1}^{i_0}$ (called strata) such that

- if $M_j \cap \overline{M_i} \neq \varnothing$, then $M_j \subset \overline{M_i}$;
- if $x \in M_j$ and $x_k \in M_i$ converge to x as $k \longrightarrow \infty$, then $T_x M_j$, the tangent space to M_j at x, is contained in the lower limit of $T_{x_k} M_i$.

Lower limit (lim inf) and upper limit (lim sup) of sets are understand in Kuratowski sense in this note. We will denote by $B_r(x)$ (respectively $\overline{B}_r(x)$) the open (respectively closed) ball in R^n with centre x and radius r.

3. Two positive results.

Theorem 3.1. *Let the compact set $K \subset R^n$ admit a regular Whitney stratification. Let $d : R^n \to R^n$ be a continuous vector field and let N_K denote the cone of limiting normals. Then the perturbed sweeping process*

$$\begin{aligned} &\dot{x}(t) \in d(x(t)) - N_K(x(t)), \\ &x(0) = x_0 \in K, \\ &x(t) \in K \end{aligned} \tag{2}$$

has a solution.

Proof. Let $\{S_i\}_{i \in I}$ be a regular Whitney stratification of K. We shall denote by S_{i_x} the stratum to which the point x belongs and by $T_x S_{i_x}$ (respectively $N_x S_{i_x}$) the tangent (respectively the normal) space at x to the smooth manifold S_{i_x}. Then we can write the drift term $d(x)$ as the sum of its tangent and normal (to S_{i_x}) component at each point $x \in K$

$$d(x) = d^T(x) + d^N(x).$$

We are going to find solution to our sweeping process restricting the right-hand side to

$$V(x) := d(x) - N_K(x) \cap \left(\|d^N(x)\| \overline{B}\right) \subset d(x) - N_K(x),$$

where $\overline{B} = \overline{B}_1(0)$. Let us prove that the multivalued mapping $V : K \to R^n$ is upper semicontinuous. As this mapping is closed valued and locally bounded in R^n, the upper semicontinuity is equivalent to the closedness of its graph. Let $\{x_m\}_{m=1}^{\infty} \subset K$ tend to $x_0 \in K$, $\{v_m\}_{m=1}^{\infty} \subset R^n$ tend to v_0 and $v_m \in V(x_m)$ for every positive integer m. It is clear that $d(x_m) \to d(x_0)$ and $d(x_0) - v_0 \in N_K(x_0)$ because of the continuity of d and upper semicontinuity of $N_K(\cdot)$, respectively.

We denote by S_j, $j \in J$ all strata with $S_{i_{x_0}} \subset \overline{S}_j$. The sequence $\{x_m\}_{m=1}^{\infty}$ can be split into finitely many subsequences (or finite sets) such that any of them is contained in one stratum S_j for some $j \in J$ or in $S_{i_{x_0}}$. Thus, without loss of generality, we may think that $\{x_m\}_{m=1}^{\infty}$ is contained in one stratum. If it is $S_{i_{x_0}}$, clearly v_0 belongs to $V(x_0)$ because of the smoothness of the manifold and the continuity of d. If $\{x_m\}_{m=1}^{\infty}$ is contained in S_j for some $j \in J$, the regularity of the stratification implies $T_{x_0} S_{i_{x_0}} \subset \liminf_{m\to\infty} T_{x_m} S_j$. This inclusion yields (c.f. Corollary 4.7 on page 113 of [10]) that

$$\|d^N(x_0)\| = \operatorname{dist}(d(x_0), T_{x_0} S_{i_{x_0}}) \geq \limsup_{m\to\infty} \operatorname{dist}(d(x_0), T_{x_m} S_j). \tag{3}$$

As

$$\operatorname{dist}(d(x_0), T_{x_m} S_j) \geq \operatorname{dist}(d(x_m), T_{x_m} S_j) - \|d(x_m) - d(x_0)\|,$$

we obtain that

$$\limsup_{m\to\infty} \operatorname{dist}(d(x_0), T_{x_m} S_j) \geq \limsup_{m\to\infty} \operatorname{dist}(d(x_m), T_{x_m} S_j) = \limsup_{m\to\infty} \|d^N(x_m)\|.$$

Now the above inequality together with (3) yields

$$\limsup_{m\to\infty} \|d^N(x_m)\| \leq \|d^N(x_0)\|,$$

thus proving that $v_0 \in V(x_0)$.

Let us consider the constrained differential inclusion

$$\begin{aligned} &\dot{x}(t) \in \overline{\text{co}}\,(V(x(t)))\,, \\ &x(0) = x_0 \in K, \\ &x(t) \in K. \end{aligned} \tag{4}$$

The right-hand side of (4) is convex compact valued. Moreover, it is upper semicontinuous. Indeed, if we assume the contrary, we can find sequences $x_m \to x_0$, $v_m \to v_0$ such that $v_m \in \overline{\text{co}}\,(V(x_m))$, $v_0 \notin \overline{\text{co}}\,(V(x_0))$. Then there exists a continuous functional $\varphi \in R^n$ and a real α with

$$\varphi(v_0) < \alpha < \varphi\left(\overline{\text{co}}\,(V(x_0))\right).$$

Now the upper semicontinuity of V implies that

$$V(x_m) \subset \{x \in R^n : \varphi(x) > \alpha\}$$

for all sufficiently large m and hence

$$v_m \in \overline{\text{co}}\,(V(x_m)) \subset \{x \in R^n : \varphi(x) > \alpha\}$$

as well, which is a contradiction. Moreover, for every $x \in K$ and every ζ, which is a proximal normal to K at x, there exists an element v of $\operatorname{co} V(x)$ such that $\langle \zeta, v\rangle \leq 0$. Indeed, $\zeta \in N_K(x)$ and $\zeta \in N_x S_{i_x}$. Take $v := d(x) - \zeta \frac{\|d^N(x)\|}{\|\zeta\|} \in V(x)$. Then

$$\langle v, \zeta\rangle = \left\langle d(x) - \zeta \frac{\|d^N(x)\|}{\|\zeta\|}, \zeta \right\rangle = \langle d(x), \zeta\rangle - \|\zeta\| \cdot \|d^N(x)\|$$
$$= \left\langle d^N(x), \zeta \right\rangle - \|\zeta\| \cdot \|d^N(x)\| \leq 0.$$

Now [11] yields the existence of a solution $x(t)$, $t \in [0,T]$ to (4).

Let $t_0 \in [0,T]$ be such that $\dot{x}(t_0)$ exists, $\dot{x}(t_0) \in \overline{\operatorname{co}}\ V(x(t_0))$ and t_0 be a cluster point of

$$\{t \in [0,T] :\ x(t) \in S_{i_{x_0}}\}.$$

Then $\dot{x}(t_0)$ must belong to $T_{x(t_0)} S_{i_{x(t_0)}}$, thus $T_{x(t_0)} S_{i_{x(t_0)}} \cap \overline{\operatorname{co}}\ V(x(t_0)) \neq \varnothing$. But the regularity of the stratification implies

$$N_{x(t_0)} S_{i_{x(t_0)}} \supset \limsup_{m\to\infty} N_{x_m} S_j$$

whenever $\{x_m\}_{m=1}^{\infty} \subset S_j$ tends to $x(t_0)$. Therefore $N_K(x(t_0)) \subset N_{x(t_0)} S_{i_{x(t_0)}}$ (because proximal normals to K at any point belong to the normal space at the same point to the stratum, to which the point belongs – see 6.8 on p. 203 in [10]). Hence

$$\overline{\operatorname{co}}\ V(x(t_0)) = d^T(x(t_0)) + \overline{\operatorname{co}}\left(d^N(x(t_0)) - N_K(x(t_0)) \cap \left(\|d^N(x(t_0))\|\overline{B}\right)\right)$$
$$\subset d^T(x(t_0)) + \left(d^N(x(t_0)) + N_{x(t_0)} S_{i_{x(t_0)}} \cap \left(\|d^N(x(t_0))\|\overline{B}\right)\right).$$

The above inclusion and $T_{x(t_0)} S_{i_{x(t_0)}} \cap \overline{\operatorname{co}}\ V(x(t_0)) \neq \varnothing$ show that

$$T_{x(t_0)} S_{i_{x(t_0)}} \cap \overline{\operatorname{co}}\ V(x(t_0)) = \{d^T(x(t_0))\}.$$

If we assume that $T_{x(t_0)} S_{i_{x(t_0)}} \cap V(x(t_0)) = \varnothing$, then the origin does not belong to

$$V(x(t_0)) - d^T(x(t_0)) = \left(d^N(x(t_0)) - N_K(x(t_0))\right) \cap \left(\overline{B}_{\|d^N(x(t_0))\|}\left(d^N(x(t_0))\right)\right)$$

and, therefore, there exists a positive real ε with

$$B_\varepsilon(0) \cap \left(V(x(t_0)) - d^T(x(t_0))\right) = \varnothing.$$

Now the uniform convexity of the ball in R^n implies the existence of a continuous linear functional φ and a positive real α such that

$$\operatorname{diam}\left\{x \in \overline{B}_{\|d^N(x(t_0))\|}\left(d^N(x(t_0))\right) :\ \varphi(x) < \alpha\right\} < \varepsilon.$$

Hence

$$V(x(t_0)) - d^T(x(t_0)) \subset \{x \in R^n : \ \varphi(x) \geq \alpha\}$$

and therefore

$$\overline{\mathrm{co}}\ V(x(t_0)) \subset d^T(x(t_0)) + \left\{x \in \overline{B}_{\|d^N(x(t_0))\|}\left(d^N(x(t_0))\right) : \ \varphi(x) \geq \alpha\right\},$$

a contradiction to $d^T(x(t_0)) \in \overline{\mathrm{co}}\ V(x(t_0))$. Hence actually

$$T_{x(t_0)} S_{i_{x(t_0)}} \cap V(x(t_0)) = \{d^T(x(t_0))\},$$

thus proving that $\dot{x}(t_0) \in V(x(t_0))$. As the set $\{t \in [0,T] : \ x(t) \in S_i\}$ may have only countably many isolated points for every $i \in I$, I is a finite set and $x(\cdot)$ is a solution of (4), we proved that $\dot{x}(t) \in V(x(t))$ for almost all $t \in [0,T]$ which concludes the proof of the theorem. □

The next theorem concerns the classical sweeping process:

Theorem 3.2. *Let the multivalued mapping $C : [0,T] \longrightarrow R^n$ be Lipschitz (with respect to the Hausdorff distance), definable in some o-minimal structure and let its values $C(t)$ be nonempty and compact. Then the sweeping process*

$$\begin{aligned} &\dot{x}(t) \in -N_{C(t)}(x(t)),\\ &x(0) = x_0 \in C(0),\\ &x(t) \in C(t), \end{aligned}$$

where $N_{C(t)}(x(t))$ denotes the cone of limiting normals to $C(t)$ at $x(t)$, has a solution.

The proof of this theorem is much longer and requires fuller use of the definability of the sets involved.

Acknowledgements. The authors are very grateful to Prof. A. Ioffe for suggesting the idea to consider the problem in the setting of some o-minimal structure. Thanks are due also to Prof. M. Krastanov for the valuable discussions and for his careful reading of the manuscript.

REFERENCES

[1] Moreau J. J., I. Sém. Anal. Convexe Montpellier, Exposé 15, 1971.
[2] Moreau J. J. J. Differential Equations, **26**, 1977, 347–374.
[3] Henry C. J. Mathematical Analysis and Application, **41**, 1971, 179–186.
[4] Bernicot F., J. Venel. `arXiv:0812.4673v3[math.CA]`.
[5] Thibault L. J. Differential Equations, **193**, 2003, 1–26.
[6] Benabdellah H. J. Differential Equations, **164**, 2000, 286–295.
[7] Colombo G., V. V. Goncharov. Set-Valued Anal., **7**, 1999, 357–374.

[8] Coste M. http://name.math.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf.
[9] Van den Dries L. Tame Topology and O-minimal Structures, Cambridge, Cambridge University Press, 1998.
[10] Rockafellar R. T., R. Wets. Variational Analysis, Springer, Grundlehren der mathematischen Wissenschaften, vol. **317**, 2nd printing, 2004.
[11] Veliov V. Set-Valued Anal., **1**, 1993, 305–317.

**Department of Mathematics and Informatics*
St. Kl. Ohridsky University of Sofia
5, J. Bourchier Blvd
1126 Sofia, Bulgaria
e-mails: bogdanmg@fmi.uni-sofia.bg
ribarska@fmi.uni-sofia.bg

***Institute of Mathematics*
and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: ribarska@fmi.uni-sofia.bg