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ON AN IMPLEMENTATION OF BLACK–SCHOLES MODEL FOR ESTIMATION OF CALL- AND PUT-OPTION VIA PROGRAMMING ENVIRONMENT MATHEMATICA

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Abstract

In this paper we propose a new module in programming environment MATHEMATICA for the Black–Scholes model taking into account the sensitivity coefficients. First we derive Black–Scholes PDE and present its explicit solution. Then we propose the sensitivity coefficients and obtain real test for them. The proposed module gives the possibility for visualization and hypersensitive analysis.

Key words: Black–Scholes model, sensitivity coefficients, programming environment MATHEMATICA

2000 Mathematics Subject Classification: 65M12, 65Y20

1. Introduction. A longstanding problem in finance was the valuation of option contracts. An option is a contract that allows the holder to buy or sell financial assets at a fixed price in the future. Is there a relationship between the price of the underlying asset, on one hand, and an option contract written on this asset? This problem was solved by F. Black and M. Scholes [1] in 1973.

Let us consider a stochastic model for the evolution of the price of the underlying asset

$$\frac{dS_t}{S_t} = \sigma dY_t + \beta dt, \tag{1}$$

where Y_t is a Brownian motion, S is the price of the underlying asset and σ, β represent respectively the volatility and mean of the returns for investing in the stock. This model is just the "continuous-time" version of the "stochastic returns" model [1].

A more generalized form of (1) is the case of coupled stochastic differential equations (SDEs), where the volatility σ is written as the square root of a variance ν

$$dS = Sbdt + S\sqrt{\nu}dY_1.$$

The variance ν is a constant in the original Black–Scholes model. Let us now assume that it follows its own SDE in the form

$$d\nu = (\omega - \Theta\nu)dt + \varepsilon\nu^{\gamma}dY_2.$$

This representation models a mean-reversion in the volatility or variance and is known as Heston model.

We shall be purposely vague about how σ and β are determined for now. Let us denote by r the prevailing short-term interest rate. Let us assume that the value V_t of a call on the stock is given by

$$V_t = C(S_t, t), \tag{2}$$

where $C(S, t)$ is a smooth function of S and t. Let an investor sell one call option and buy Δ shares of the underlying asset at time t. The change in the value of his holdings over the interval $(t, t + dt)$ is

$$(-V_{t+dt} + \Delta S_{t+dt}) - (-V_t + \Delta S_t) = -dV_t + \Delta dS_t. \tag{3}$$

Combining (1) and (2) and applying Ito's formula [2,3], we can express the variation of the portfolio in terms of the variation of the price of the underlying asset

$$dV_t = C_S(S_t, t)dS_t + C_t(S_t, t)dt + \frac{1}{2}\sigma^2 S^2 C_{SS}(S_t, t)dt \tag{4}$$

to leading order in dt. Substituting this equation into (3), we obtain the following equation for the change in the portfolio value

$$(-C_S(S_t, t) + \Delta)dS_t - (C_t(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 C_{SS}(S_t, t))dt. \tag{5}$$

If the number of shares held in the portfolio was

$$\Delta = C_S(S_t, t)$$

then the dx_t term would vanish in equation (5), rending the return of the portfolio non-volatile over the period of time $(t, t + dt)$ (to leading order in dt).

Since the value of the option-stock portfolio at time t is $-C(S_t, t) + \Delta S_t = -C(S_t, t) + S_t C_S(S_t, t)$, the absence of arbitrage implies that

$$C_t + \frac{\sigma^2}{2} S^2 C_{SS} = r(C - SC_S)$$

or

$$C_t + \frac{\sigma^2}{2} S^2 C_{SS} + rSC_S - rC = 0. \tag{6}$$

This is the Black–Scholes PDE. To determine the function $C(S,T)$, we must specify boundary conditions. In the case of a call with expiration date T we have

$$C(S,T) = (S - X)^+ = \max(S - X), \tag{7}$$

where X is the strike price (z^+ represents the positive part of z). Indeed, if $S_T \leq X$, the option is worthless, and if $S_T > X$, the holder of the call can buy the underlying asset for X dollars and sell it at a market price, making a profit of $S_T - X$.

For an European-style call (which can be exercised only at date T), $C(S,t)$ is determined by solving the Cauchy problem for the Black–Scholes PDE with condition (7). To value American options, the idea is that we should look for a function $C(S,t)$ that satisfies the Black–Scholes equation in the regions of the (S,t)-plane where an option should not be exercised and provide additional boundary conditions along the region corresponding to price levels where the option should be exercised. One way to arrive at this region is to impose the additional condition on option prices that should hold in the case of American-style options

$$\begin{aligned} C(S,t) &\geq (S - X)^+ \qquad \text{(calls)},\\ P(S,t) &\geq (X - S)^+ \qquad \text{(puts)}, \end{aligned}$$

since the option is worth at least as much as what would get exercising it immediately. This constraints give rise to an obstacle problem, or differential inequality, for the Black–Scholes equation which can be solved numerically.

The Black–Scholes PDE has a fundamental probabilistic interpretation. The correspondence between PDEs and the probabilities via Fokker–Plank formalism yields

$$C(S_t, t) = \mathbf{E}\Big\{ \sum_{i:t<T_i} e^{-r(T_i - t)} F(S_{T_i}) | I_t \Big\},$$

where $\mathbf{E}\{\cdot|I_t\}$ represents the conditional expectation, $T_1 < T_2 < \cdots < T_N$ are different dates for the series of cash-flows represented by $F_i(S_{T_i})$, $i = 1, 2, \ldots, N$. S_t is the diffusion process governed by the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma dY_t + rdt.$$

In the case of a call-option, the explicit solution of Black–Scholes equation (6) is

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2), \tag{8}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}, \end{aligned} \tag{9}$$

where C_0 is the value of a call on the stock; S_0 is the evolution of the price of the underlying asset; $N(d)$ is the standard normal commutative distribution; X is the strike price; r is the interest rate; T is the expiration date (in years); σ is the volatility.

In the case of a put-option we have the following solution:

$$P_0 = C_0 + Xe^{-rT} - S_0. \tag{10}$$

We want to point out that the original formula (1)–(3) can be used in practice for the "current date" t in the following way:

$$C_0 = S_0 N(d_1) - Xe^{-r(T-t)} N(d_2), \tag{11}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{12}$$

$$\begin{aligned} P_0 &= C_0 + Xe^{-r(T-t)} - S_0 \\ &= S_0 N(d_1) - Xe^{-r(T-t)} N(d_2) + Xe^{-r(T-t)} - S_0 \\ &= Xe^{-r(T-t)}\left(1 - N(d_2)\right) - S_0\left(1 - N(d_1)\right) \\ &= Xe^{-r(T-t)} N(-d_2) - S_0 N(-d_1). \end{aligned} \tag{13}$$

In the field of applied financial mathematics the following coefficients are used in order to estimate market sensitivity of the options or other financial instruments (see [4]):

1. Coefficient *Delta*

$$\delta_{C_0} = N(d_1), \qquad \delta_{P_0} = N(d_1) - 1. \tag{14}$$

2. Coefficient *Gamma*

$$\gamma_{C_0} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}, \qquad \gamma_{P_0} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}, \tag{15}$$

where

$$N'(d_1) = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}.$$

3. Coefficient *Vega*

$$v_{C_0} = S_0 N'(d_1)\sqrt{T}, \qquad v_{P_0} = S_0 N'(d_1)\sqrt{T}. \tag{16}$$

4. Coefficient *Theta*

$$\theta_{C_0} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2), \ \ \theta_{P_0} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rXe^{-rT}N(-d_2). \tag{17}$$

5. Coefficient *Rho*

$$\rho_{C_0} = XTe^{-rT}N(d_2), \qquad \rho_{P_0} = -XTe^{-rT}N(-d_2). \tag{18}$$

In the case of the "current date" the t coefficients for the market sensitivity are the following:

1′. Coefficient *Delta*

$$\delta_{C_0} = N(d_1), \quad \delta_{P_0} = N(d_1) - 1. \tag{19}$$

2′. Coefficient *Gamma*

$$\gamma_{C_0} = \frac{N'(d_1)}{S_0\sigma\sqrt{T-t}}, \qquad \gamma_{P_0} = \frac{N'(d_1)}{S_0\sigma\sqrt{T-t}}, \tag{20}$$

where

$$N'(d_1) = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}.$$

3′. Coefficient *Vega*

$$v_{C_0} = S_0 N'(d_1)\sqrt{T-t}, \qquad v_{P_0} = S_0 N'(d_1)\sqrt{T-t}. \tag{21}$$

4′. Coefficient *Theta*

$$\begin{aligned} \theta_{C_0} &= -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T-t}} - rXe^{-r(T-t)}N(d_2), \\ \theta_{P_0} &= -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T-t}} + rXe^{-r(T-t)}N(-d_2). \end{aligned} \tag{22}$$

5′. Coefficient *Rho*

$$\rho_{C_0} = X(T-t)e^{-r(T-t)}N(d_2), \qquad \rho_{P_0} = -X(T-t)e^{-r(T-t)}N(-d_2). \tag{23}$$

Remark 1. Black–Scholes model is approximately accurate due to some suggestions [5]. For this purpose, we are sure only for four parameters C, X, T and r. Recall that the parameters that enter the Black–Scholes formula are: (i) the exercise price, or strike price X; (ii) the expiration date T; (iii) the price of the underlying asset C; (iv) the interest rate r; (v) the volatility σ. Of these five parameters, the first four are observable at any given time. In contrast, the volatility of the underlying asset is not directly observable. For each value of the volatility we obtain a different theoretical option value. Consequently, it is easy to show that to each possible option value corresponds a unique volatility parameter. This is a consequence of the fact that the Black–Scholes option premium is a strictly increasing function of σ. The implied volatility of a traded call is, by definition, the value of σ that solves the equation

$$C(S, t; X, T, r, \sigma) = \text{market price of the call},$$

where the left-hand side represents the Black–Scholes theoretical value, with the same definition applied to puts.

Some modules in programming environment MATHEMATICA can be upgraded with the coefficients mentioned above.

In this paper our aim is to develop a user module in MATHEMATICA as an implication of the Black–Scholes model. This includes:

Parameters' input:

1. Parameter S_0 is the current price of the underlying asset – 81;
2. Parameter X is the strike price of the option – 80;
3. Parameter r is the interest rate under the year basis of unrisky asset with maturity equal to the date of option validity – 0.06;
4. Parameter T is the expiration date (in years) – 0.166;
5. Parameter σ is the volatility of the underlying asset – 0.3.

Current parameter:

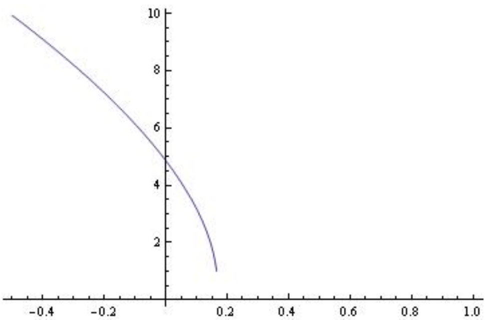
t – for the calculation of call and put options at time t_i, $i = 1, 2, \ldots$

Procedures:

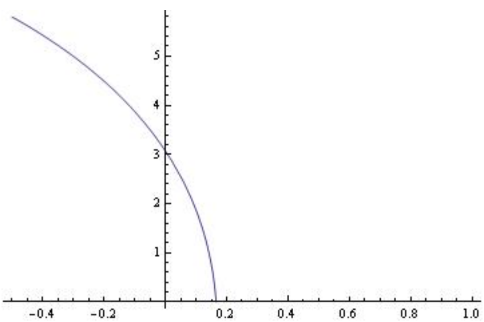
1. Repeatedly addressed the operator of programming environment MATHEMATICA for calculation of normal distribution at points;
2. Calculation of the values of call and put options and visualization;
3. Calculation of the coefficients of sensitivity: Deltac, Deltap, Rhoc, Rhop, Vegas, Thetac, Thetap and visualization.

Below the reader can see the test provided on our control example for Black–Scholes model, realized in software packages of the programmes Excel, Matlab, Maple, etc. [6–9].

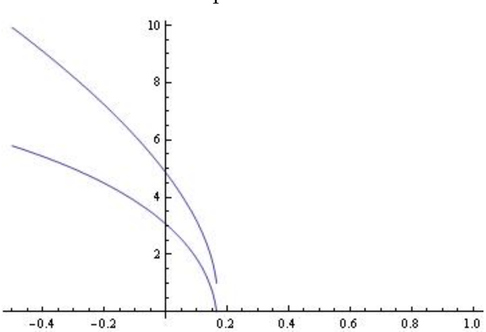
Current price of the underlying asset S0 = 81
Strike price of the option X = 80
Interest rate under the year basis of unrisky asset with maturity equal to the date of option validity r = 0.06
Expiration date (in years) T = 0.166
Volatility of the underlying asset $\sigma = 0.3$
Date for which expected values of the Call and Put options t0 = 0
Current value of the Call option:
C0 = 4.86522
The Call option



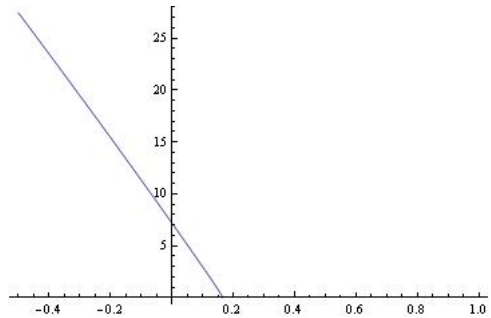
Current value of the Put option:
P0 = 3.07237
The Put option



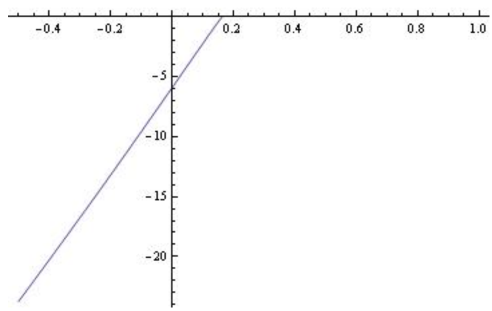
The Call and Put options



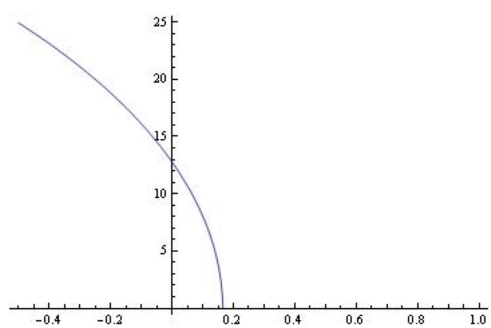
Value of the coefficient Deltac:
Deltac = 0.596475
Value of the coefficient Deltap:
Deltap = –0.403525
Value of the coefficient Rhoc:
Rhoc = 7.21258
The function of Rhoc



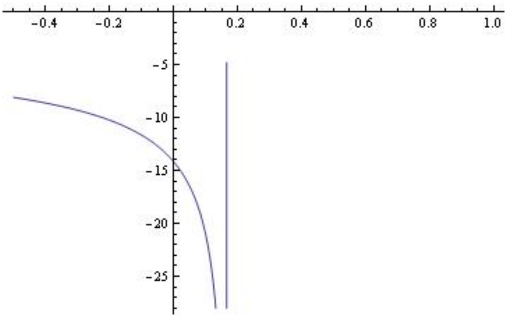
Value of the coefficient Rhop:
Rhop = –5.93581
The function of Rhop



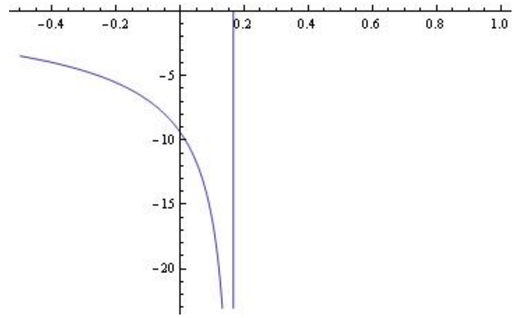
Value of the coefficient Vegac:
Vegac = 12.779
The function of Vegac



Value of the coefficient Thetac:
Thetac = –14.1542
The function of Thetac



Value of the coefficient Thetap:
Thetap = –9.4018
The function of Thetap



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