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OPTIMAL EMBEDDINGS OF GENERALIZED SOBOLEV SPACES IN HÖLDER–ZYGMUND SPACES

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Abstract

We prove optimal embeddings of the generalized Sobolev spaces W^kE, where E is a rearrangement invariant function space, into the generalized Hölder–Zygmund space $\mathcal{C}H$ generated by a function space H.

Key words: generalized Sobolev spaces, rearrangement invariant function spaces, Hölder–Zygmund spaces, extrapolation

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1. Introduction. The classical Sobolev space W_p^k, $1 \leq p < \infty$, consists of all locally integrable functions f, defined on $\mathbf{R}^n$, $n \geq 1$, with the Lebesgue measure, such that the following norm is finite: $\|f\|_{W_p^k} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_p$, where $\|f\|_p$ stands for the L^p-norm. In investigating the regularity of the function $f \in W_p^k$ we may assume, without any loss of generality, that $f \in L^1(\Omega)$, Ω is a domain in $\mathbf{R}^n$ and f is zero outside Ω. For simplicity we suppose that the Lebesgue measure of Ω equals one and that the origin lies in Ω. It is well known that in the supercritical case $k > n/p$,

$$W_p^k \hookrightarrow \mathcal{C}^{k-n/p},\ k > n/p, \tag{1.1}$$

where $\mathcal{C}^\gamma$, $\gamma > 0$, is the Hölder–Zygmund space (see [1]). In the critical case $k = n/p$ the function $f \in W_p^k$ may not be even continuous. The result (1.1) is not optimal. We prove that the optimal one is obtained if in (1.1) L^p is replaced by the Marcinkiewicz space $L^{p,\infty}$. We prove similar optimal results, when $L^{p,\infty}$

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is replaced by a more general rearrangement invariant space E. The Sobolev space W^kE is defined with a quasi-norm $\|f\|_{W^kE} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_E$. More precisely, we consider quasi-normed rearrangement invariant spaces E, consisting of functions $f \in L^1(\Omega)$, such that the quasi-norm $\|f\|_E \approx \rho_E(f^*) < \infty$, where ρ_E is a monotone quasi-norm, defined on M^+ with values in $[0,\infty]$ and M^+ is the cone of all locally integrable functions $g \geq 0$ on $(0,1)$ with the Lebesgue measure. Monotonicity means that $g_1 \leq g_2$ implies $\rho_E(g_1) \leq \rho_E(g_2)$. We suppose that $L^\infty(\Omega) \hookrightarrow E \hookrightarrow L^1(\Omega)$, which means continuous embeddings. Here f^* is the decreasing rearrangement of f (see [2]). Let α_E, β_E be the Boyd indices of E (see [2]). For example, if $E = L^p$, then $\alpha_E = \beta_E = 1/p$ and the condition $k/n \geq 1/p$ means $k/n \geq \alpha_E > 0$. Note that for $k > n$ this is always satisfactory. For these reasons we suppose that for the general E, $0 < \alpha_E = \beta_E \leq 1$ and the case $\min(k,n)/n > \alpha_E$ is called supercritical, while the case $\min(k,n)/n = \alpha_E$ – critical. In the supercritical case the function $f \in W^kE$ is always continuous, while the spaces in the critical case $\alpha_E = k/n$, $k < n$, can be divided into two subclasses: in the first subclass the functions $f \in W^kE$ may not be continuous – then the target space is rearrangement invariant, while these functions in the second subclass are continuous and the target space is the generalized Hölder–Zygmund space $\mathcal{C}H$. The separating space for these two subclasses is given by the Lorentz space $L^{n/k,1}$, $k < n$. If $k \geq n$, then W^kE consists of continuous functions.

The main goal is to prove optimal embeddings of the Sobolev space W^kE into the generalized Hölder–Zygmund space $\mathcal{C}H$. First we prove that this embedding for $k \leq n$ is equivalent to the continuity of the operator $R_kg(t) = \int_0^t u^{k/n-1}g(u)du$. The case $k > n$ is reduced to the continuity of R_n by using the lifting principle [1]. Moreover, if, for example, $k \leq n$, then in the supercritical case, we can replace R_k by the operator of multiplication $t^{k/n}g(t)$. This implies a very simple characterization of both optimal target space H and optimal domain space E. Namely, the quasi-norm in the optimal target space $H(E)$ is given by $\rho_E(t^{-k/n}g(t))$ and the quasi-norm in the optimal domain space $E(H)$ is given by $\rho_H(t^{k/n}g(t))$. Note that we do not require ρ_E to be a rearrangement invariant. In the critical case, the formula for the optimal target space is more complicated. In some cases it can be simplified. To this end, we apply the Σ^q-method of extrapolation [3] from the supercritical case. As a byproduct, we also characterize the embedding $W^kE \hookrightarrow C^j$, $j < k$, where C^j consists of all functions with bounded and uniformly continuous derivatives up to order j. Namely, this is equivalent to the embedding $E \hookrightarrow L^{n/(k-j),1}$ if $k \leq n$. The embedding $W^{n+j}E \hookrightarrow C^j$ is always true since $W^nE \hookrightarrow W_1^n \hookrightarrow C^0$.

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$. Let E be a quasi-normed rearrangement invariant space. There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1+g_2) \leq \rho_p^p(g_1)+\rho_p^p(g_2)$ for some

$p \in (0, 1]$ that depends only on the space E (see [4]). We say that the quasi-norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\left(\sum g_j\right) \lesssim \sum \rho_p^p(g_j),\ g_j \in M^+.$$

Usually we apply this inequality for functions $g_j \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let $g_u(t) = g(t/u)$ if $t < u$ and $g_u(t) = 0$ if $t \geq u$, where $0 < t < 1,\ g \in M^+$, and let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+\right\},\ u > 0$$

be the dilation function generated by ρ_E. Suppose that it is finite. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \text{ and } \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$ hence $0 \leq \alpha_E \leq \beta_E$. We suppose that $0 < \alpha_E = \beta_E \leq 1$.

If $\beta_E < 1$, we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(f^{**})$, where $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$. In particular, $\|f\|_E \approx \rho_E(f^{**})$ if $\beta_E < 1$. For example, consider Gamma spaces $E = \Gamma^q(w)$, $0 < q \leq \infty$, w – positive weight, i.e. a positive function from M^+, with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_E(f^*)$, $\rho_E(g) := \rho_{w,q}(\int_0^1 g(tu)du)$, where

$$\rho_{w,q}(g) := \left(\int_0^1 [g(t)w(t)]^q dt/t\right)^{1/q},\quad g \in M^+, \tag{1.2}$$

and

$$\left(\int_0^1 w^q(t)dt/t\right)^{1/q} < \infty.$$

Then $L^\infty(\Omega) \hookrightarrow \Gamma^q(w) \hookrightarrow L^1(\Omega)$. If $w(t) = t^{1/p}$, $1 < p < \infty$, we write as usual $L^{p,q}$ instead of $\Gamma^q(t^{1/p})$. Consider also the classical Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda^q_w} := \rho_{w,q}(f^*) < \infty$, $w(2t) \approx w(t)$. We suppose that $L^\infty(\Omega) \hookrightarrow \Lambda^q(w) \hookrightarrow L^1(\Omega)$.

Note that if $E = \Lambda^q(t^\alpha w)$, $0 < \alpha \leq 1$, where w is slowly varying, then $\alpha_E = \beta_E = \alpha$. Recall that $w \in M^+$ is slowly varying if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to an increasing function, and the function $t^{-\varepsilon}w(t)$ is equivalent to a decreasing function.

In order to introduce the Hölder–Zygmund class of spaces, we denote the modulus of continuity of order m by

$$\omega^m(t, f) = \sup_{|h|\leq t}\sup_{x\in\mathbf{R}^n} |\Delta_h^m f(x)|.$$

where $\Delta_h^m f$ are the usual iterated differences of f. When $m=1$ we simply write $\omega(t,f)$. Let H be a quasi-normed space of locally integrable functions on the interval $(0,1)$ with the Lebesgue measure, continuously embedded in $L^\infty(0,1)$ and $\|g\|_H=\rho_H(|g|)$, where ρ_H is a monotone quasi-norm on M^+ which satisfies Minkowski's inequality. The dilation function generated by ρ_H is given by

$$h_H(u)=\sup\left\{\frac{\rho_H(\tilde g_u)}{\rho_H(g)}: g\in M_m\right\},$$

where $\tilde g_u(t)=g(ut)$ if $ut<1$, $\tilde g_u(t)=g(1)$ if $ut\ge 1$, $0<t<1$, and

$$M_m:=\{g\in M^+: t^{-m/n}g(t) \text{ is decreasing}\},\ m>k.$$

The choice of the space M_m is motivated by the fact that $\omega^m(t^{1/n},f)$ is equivalent to a function $g\in M_m$. The function $h_H(u)$ is sub-multiplicative and $u^{-m/n}h_H(u)$ is decreasing and $h_H(1)=1,\ h_H(u)h_H(1/u)\ge 1$. Suppose that h_H is finite. Then the Boyd indices of H are well-defined

$$\alpha_H=\sup_{0<t<1}\frac{\log h_H(t)}{\log t}\quad\text{and}\quad \beta_H=\inf_{1<t<\infty}\frac{\log h_H(t)}{\log t},$$

and they satisfy $\alpha_H\le\beta_H\le m/n$. In what follows, we suppose that $0\le\alpha_H=\beta_H\le m/n$. For example, let $H=L_*^q(b(t)t^{-\gamma/n})$. Here $0\le\gamma\le m$ and b is a slowly varying function, and $L_*^q(w)$, or simply L_*^q if $w=1$, is the weighted Lebesgue space with a quasi-norm $\|g\|_{L_*^q(w)}=\rho_{w,q}(|g|)$, where $\rho_{w,q}$ is given by (1.2). It turns out that $\alpha_H=\beta_H=\gamma/n$.

Definition 1.1. *Let $j=0,1,\dots$ and let C^j stand for the space of all functions f, defined on $\mathbf{R}^n$, that have bounded and uniformly continuous derivatives up to the order j, normed by $\|f\|_{C^j}=\sup\sum_{l=0}^j|P^lf(x)|$, where $P^lf(x)=\sum_{|\nu|=l}D^\nu f(x)$.*

- *If $j/n<\alpha_H<(j+1)/n$ for $j\ge 1$ or $0\le\alpha_H<1/n$ for $j=0$, then $\mathcal{C}H$ is formed by all functions f in C^j having a finite quasi-norm*

$$\|f\|_{\mathcal{C}H}=\|f\|_{C^j}+\rho_H(\chi_{(0,1)}(t)t^{j/n}\omega(t^{1/n},P^jf)).$$

- *If $\alpha_H=(j+1)/n$, then $\mathcal{C}H$ consists of all functions f in C^j having a finite quasi-norm*

$$\|f\|_{\mathcal{C}H}=\|f\|_{C^j}+\rho_H(\chi_{(0,1)}(t)t^{j/n}\omega^2(t^{1/n},P^jf)).$$

Here $\chi_{(a,b)}$, $0<a<b<\infty$, is the characteristic function of the interval (a,b).

In particular, if $H=L^\infty(t^{-\gamma/n})$, $\gamma>0$, then $\mathcal{C}H$ coincides with the usual Hölder–Zygmund space $\mathcal{C}^\gamma$ (see [1]). Also, if $H=L^\infty$, then $\mathcal{C}H=C^0$.

We shall use the following equivalent quasi-norm.

Theorem 1.2 (equivalence)**.** *Let* $0 \le \alpha_H = \beta_H < m/n$*. If* $\rho_H(\chi_{(0,1)}(t)t^{\alpha}) < \infty$ *for* $\alpha > \alpha_H$*, then for all such* m*,*

$$\|f\|_{\mathcal{C}H} \approx \|f\|_{C^0} + \rho_H(\chi_{(0,1)}(t)\omega^m(t^{1/n}, f)).$$

Theorem 1.3 (lifting principle)**.** *Let* $\alpha_H > 0$ *and let* $\rho_{H_j}(g) := \rho_H(t^{-j/n}g(t))$*,* $j \ge 1$*. Then*

$$\|f\|_{\mathcal{C}H_j} \approx \|\mathcal{D}^j f\|_{\mathcal{C}H},\ \mathcal{D}^j = \sum_{|\alpha|\le j} D^{\alpha}.$$

It will be convenient to introduce the classes of the domain and target quasi-norms, where the optimality is investigated. Let N_d consist of all domain quasi-norms ρ_E that are monotone, satisfy Minkowski's inequality, $0 < \alpha_E = \beta_E \le \min(k,n)/n$, $\rho_E(\chi_{(0,1)}t^{-\alpha}) < \infty$ if $\alpha < \alpha_E$ and the condition (2.2) below for $k \le n$, and $E \hookrightarrow L^1$ for $k > n$. Let N_t consist of all target quasi-norms ρ_H that are monotone, satisfy Minkowski's inequality, $0 \le \alpha_H = \beta_H < \min(k,n)/n$, $\rho_H(\chi_{(0,1)}(t)t^{\alpha}) < \infty$ if $\alpha > \alpha_H$ and $\sup_{0<t<1} g(t) \lesssim \rho_H(g)$.

We use the following definitions.

Definition 1.4 (admissible couple)**.** *We say that the couple* $\rho_E \in N_d$*,* $\rho_H \in N_t$ *is admissible if* $W^kE \hookrightarrow \mathcal{C}H$ *when* $k \le n$*, and if* $\mathcal{D}^j(W^{n+j}E) \hookrightarrow \mathcal{C}H$ *for* $j \ge 1$*. Moreover,* ρ_E (E) *is called domain quasi-norm* (*domain space*)*, and* ρ_H (H) *is called target quasi-norm* (*target space*)*.*

Let

$$M_0 = \{g \in M^+ : g \text{ is increasing, } t^{-\min(k,n)/n}g(t) \text{ is decreasing and } g(+0) = 0\}.$$

The choice of M_0 is motivated by the fact that $\omega^{\min(k,n)}(t^{1/n}, f)$, $0 < t < 1$, is equivalent to a function $g \in M_0$ if $f \in W^{\min(k,n)}E \hookrightarrow C^0$.

Definition 1.5 (optimal target quasi-norm)**.** *Given the domain quasi-norm* $\rho_E \in N_d$*, the optimal target quasi-norm, denoted by* $\rho_{H(E)}$*, is the strongest target quasi-norm, i.e.* $\rho_H(g) \lesssim \rho_{H(E)}(g)$*,* $g \in M_0$ *for any target quasi-norm* $\rho_H \in N_t$ *such that the couple* ρ_E, ρ_H *is admissible. Since* $\mathcal{C}H(E) \hookrightarrow \mathcal{C}H$*, we call* $\mathcal{C}H(E)$ *the optimal Hölder–Zygmund space.*

Definition 1.6 (optimal domain quasi-norm)**.** *Given the target quasi-norm* $\rho_H \in N_t$*, the optimal domain quasi-norm, denoted by* $\rho_{E(H)}$*, is the weakest domain quasi-norm, i.e.* $\rho_{E(H)}(f^*) \lesssim \rho_E(f^*)$*,* $f \in L^1(\Omega)$*, for any domain quasi-norm* $\rho_E \in N_d$ *such that the couple* ρ_E, ρ_H *is admissible.*

Definition 1.7 (optimal couple)**.** *The admissible couple* $\rho_E \in N_d$*,* $\rho_H \in N_t$ *is said to be optimal if both* ρ_E *and* ρ_H *are optimal.*

The problem of the optimal target rearrangement invariant space for potential type operators is considered in [5] by using L^p-capacities. The problem of the mapping properties of the Riesz potential in optimal couples of rearrangement invariant spaces is treated in [6–8]. The characterization of the continuous embedding of the generalized Bessel potential spaces into the generalized Hölder–Zygmund spaces $\mathcal{C}H$, when H is a weighted Lebesgue space, is given in [9]. Our method is different and more general and it could be applied for Bessel potential spaces as well.

2. Admissible couples. Here we give a characterization of all admissible couples $\rho_E \in N_d$, $\rho_H \in N_t$.

Theorem 2.1 (main estimate)**.** *Let $f \in W_1^k$ and $k \leq n$. Then*

$$\omega^k(t^{1/n}, f) \lesssim \int_0^t u^{k/n-1}(\mathcal{D}^k f)^*(u)du.$$

Theorem 2.2. *A necessary and sufficient condition for the embedding $W^kE \hookrightarrow C^0$, $k \leq n$, is the following one*

$$\int_0^1 t^{k/n-1}g(t)dt \lesssim \rho_E(g),\ g \in D_1 := \{g \in M^+ : g \textit{ is decreasing function}\}.$$

Remark 2.3. $W^kE \hookrightarrow C^j$, $j < k \leq n$, *if and only if* $E \hookrightarrow L^{n/(k-j),1}$.

Theorem 2.4. *The couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible if and only if*

$$\rho_H(R_{\min(k,n)}g) \lesssim \rho_E(g),\ g \in D_1,\ R_kg(t) := \int_0^t u^{k/n-1}g(u)du.$$

Theorem 2.5. *Let $\alpha_H > 0$. Then the couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible for $k > n$ if and only if*

$$W^{n+j}E \hookrightarrow \mathcal{C}H_j,\ \rho_{H_j}(g) := \rho_H(t^{-j/n}g(t)),\ j \geq 1. \tag{2.1}$$

Moreover, (2.1) *is equivalent to* $\rho_{H_j}(R_{n+j}g) \lesssim \rho_E(g),\ g \in D_1$.

3. Optimal quasi-norms. Here we give a characterization of the optimal domain and optimal target quasi-norms.

3.1. Optimal domain quasi-norms. We can construct an optimal domain quasi-norm $\rho_{E(H)}$ by Theorem 2.1 as follows. For a given target quasi-norm $\rho_H \in N_t$, we set

$$\rho_{E(H)}(g) := \rho_H(R_{\min(k,n)}g),\ g \in M^+.$$

Note that $\alpha_{E(H)} = \beta_{E(H)} = \min(k,n)/n - \alpha_H$.

Theorem 3.1. *The quasi-norm $\rho_{E(H)}$ belongs to N_d, the couple $\rho_{E(H)}, \rho_H$ is admissible, the domain quasi-norm $\rho_{E(H)}$ is optimal. Moreover, the target quasi-norm ρ_H is also optimal and*

$$\rho_{E(H)}(g) \approx \rho_H(t^{\min(k,n)/n}g),\ g \in D_1 \ \textit{ if } \ \alpha_H > 0.$$

Example 3.2. *Consider the space $H = L^1_*(v)$, where $\rho_H(g) = \int_0^1 v(t)g(t)dt/t$ and $\rho_H \in N_t$. Using Theorem 3.1, we can construct an optimal domain E, where*

$$\rho_E(g) = \rho_H(R_{\min(k,n)}g) = \int_0^1 t^{\min(k,n)/n} w(t)g(t)dt/t$$

and $w(t) = \int_t^1 v(u)du/u$. Hence $E = \Lambda^1(t^{\min(k,n)/n}w)$ and this couple is optimal. Also $\alpha_E = \beta_E = \min(k,n)/n$ if v is slowly varying. Note that if $k \geq n$, then $E = \Lambda^1(tw) = \Gamma^1(tv)$.

Example 3.3. *Let $H = L^\infty(v)$, where $\rho_H(g) = \sup v(t)g(t)$ and $\rho_H \in N_t$ and let*

$$\rho_E(g) = \sup v(t) \int_0^t u^{\min(k,n)/n} g(u)du/u.$$

Then by Theorem 3.1, the domain E is optimal and the couple is optimal. In particular, the couple $L^{n/\min(k,n),1}$, C^0 is optimal. If $k = n + j$, $j \geq 0$, this means that the embedding $W_1^{n+j} \hookrightarrow C^j$ is optimal.

Example 3.4. *Let H be as in the previous example. Since*

$$\rho_E(g) \leq \sup t^{\min(k,n)/n} w(t)g(t), \ \frac{1}{v(t)} = \int_0^t \frac{1}{w(u)} \frac{du}{u},$$

it follows that the couple $E = \Lambda^\infty(t^{\min(k,n)/n}w), H = L^\infty_*(v)$ is admissible. We prove that ρ_H is optimal.

Example 3.5 (case $k \geq n$)**.** *Let $H = L^q_*(v)$, $0 < q \leq \infty$, where $\rho_H(g) = \left(\int_0^1 [v(t)g(t)]^q dt/t\right)^{1/q}$ and $\rho_H \in N_t$ and let $k \geq n$. Then the couple $E = \Gamma^q(tv)$, H is optimal. Also $\alpha_E = \beta_E = 1$ if v is slowly varying.*

3.2. Optimal target quasi-norms. For a given domain quasi-norm $\rho_E \in N_d$, we set

$$\rho_{H(E)}(g) := \inf\{\rho_E(h) : g \leq R_{\min(k,n)}h, \ h \in D_1\}, \ g \in M^+. \tag{3.1}$$

Note that $\alpha_{H(E)} = \beta_{H(E)} = \min(k,n)/n - \alpha_E$.

Theorem 3.6. *The target quasi-norm $\rho_{H(E)}$ belongs to N_t, the couple ρ_E, $\rho_{H(E)}$ is admissible and the target quasi-norm is optimal.*

Theorem 3.7 (supercritical case)**.** *If $\alpha_E < \min(k,n)/n$, then*

$$\rho_{H(E)}(g) \approx \rho_E(t^{-\min(k,n)/n}g(t)), \ g \in M_{\min(k,n)}.$$

Moreover, the couple $\rho_E, \rho_{H(E)}$ is optimal.

Example 3.8. *Consider the space $E = \Lambda^q(w)$, $0 < q \leq \infty$, $\min(k,n)/n > \beta_E = \alpha_E > 0$. Then by Theorem 3.7 the couple E, H, $H = L^q_*(t^{-\min(k,n)/n}w)$ is optimal. In particular, using also Theorem 2.5, the embedding $W^k L^{p,\infty} \hookrightarrow \mathcal{C}^{k-n/p}$, $k > n/p$, $1 < p < \infty$, is optimal.*

In the critical case we do not know how to simplify the optimal target quasi-norm, defined in (3.1). Instead, we can construct a large class of domain quasi-norms and the corresponding optimal target quasi-norms by using extrapolation from the supercritical case.

Theorem 3.9. *Let* $E = \Lambda^q(t^{k/n}c(t)(1-\ln t))$, $k < n$, c *– slowly varying weight,* $c(+0) = \infty$, $c(t^2) \approx c(t)$, $0 < q \le \infty$, $H = L^q_*(c)$. *We suppose that* $\rho_E \in N_d$ *and* $\rho_H \in N_t$. *Then this couple is admissible and the target quasi-norm is optimal.*

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