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AUTOMORPHISMS IN THE COMMUTANT OF A GENERAL OPERATOR OF INTEGRATION TYPE

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Abstract

The operator $M_p y(z) = \sum_{k=0}^{\infty} \left[\frac{y^{(k)}(z)}{k!}\right]_{z=0} b_k z^{k+p}$, where $p \geq 0$ is an integer and b_k are arbitrary complex numbers, is considered in the space A_R of the functions analytic in the disc $\{|z| < R\}$. In [5] a power series description of the commutant of M_p was given by the author. Here we continue these investigations showing cases when the operators of the commutant are or are not isomorphisms.

Key words: commutant, isomorphism
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1. Introduction. Let A_R be the space of the functions of a complex variable z, analytic in the disc $\{|z| < R\}$, with the topology of the uniform convergence on the compact subsets. Here we consider the linear operator

(1) $$M_p y(z) = \sum_{k=0}^{\infty} \left[\frac{y^{(k)}(z)}{k!}\right]_{z=0} b_k z^{k+p}, \quad p \in \mathbb{N}_0, \quad b_k \in \mathbb{C},\ b_k \neq 0.$$

Denoting the coefficients $\left[\frac{y^{(k)}(z)}{k!}\right]_{z=0}$ by a_k and assuming that M_p is a continuous operator, M_p acts on $y(z) = \sum_{k=0}^{\infty} a_k z^k$ as

(2) $$M_p y(z) = M_p\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=0}^{\infty} a_k b_k z^{k+p}, \quad b_k \neq 0, k = 0, 1, 2, \dots.$$

The action of the operator M_p on a single power z^k is given by

(3) $$M_p z^k = b_k z^{k+p},\ b_k \neq 0 \ \text{– arbitrary.}$$

This operator can be considered as a generalization of various operators of integration type which preserve or increase the powers by a fixed number $p \in \mathbb{N}_0$. If $y(z) = \sum_{k=0}^{\infty} a_k z^k$, such an operator is the so-called generalized Gel'fond–Leont'ev integration operator [4]

$$I_{\varphi}^{(p)} y(z) = \sum_{k=0}^{\infty} a_k \frac{\varphi_{k+p}}{\varphi_k} z^{k+p},$$

generated by an entire function $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k$ of order $\rho > 0$ and type σ, such that $\lim_{k\to\infty} k^{1/\rho} = (\sigma e \rho)^{1/\rho}$. In the case of an operator (1) with $p = 1$, the generating function is taken so that $b_k = \dfrac{\varphi_{k+1}}{\varphi_k}$. For example, the classical integration operator (of order 1), $M_p = M_1 = l^1$ has coefficients $b_k = \dfrac{1}{k+1} = \dfrac{k!}{(k+1)!}$ and the generating function is $\varphi(z) = e^z$. For l^p, the integration of order p, one has $b_k = \dfrac{k!}{(k+p)!}$. A convolutional representation of the commutants of such operators is given by Dimovski in [2] (see §2.4.3, 2.4.4). Commutants of integration operators were considered in many papers as in [1], etc.

The operator M_p is also an Hadamard product

$$M_p y(z) = z^p[y(z) \circ b(z)] = z^p \left[\left(\sum_{k=0}^{\infty} a_k z^k \right) \circ \left(\sum_{k=0}^{\infty} b_k z^k \right) \right] = \sum_{k=0}^{\infty} a_k b_k z^{k+p},$$

and to consider such operator as a generalized integration, one has to assume $p \geq 0$ and $\lim_{k\to\infty} b_k = 0$ (see S. G. Samko, A. A. Kilbas, O. I. Marichev [8], Sect. 22).

Commutants of different operators are important and are considered by many mathematicians, not only earlier, as in [3], but also nowadays, as in [7]. The author has also several papers related to commutants, from which we shall mention only [6], devoted to generalizations of the Hardy–Littlewood operator $H_{m,n} y(z) = \dfrac{1}{z^n} \displaystyle\int_0^z t^m y(t) dt$, which is of type (1), (2) or (3).

Let us give some definitions:

Definition 1. The set C_{M_p} of all continuous linear operators $L : A_R \to A_R$ commuting with M_p, i.e. such that $L M_p = M_p L$, is said to be the commutant of M_p.

In this paper we include without proofs some results from our previous paper [5] giving a power series description of the commutant C_{M_p} of the operator M_p defined by (1), (2) or (3). Our aim here is to propose additional conditions under which an operator of the commutant C_{M_p} is an isomorphism of A_R and to show examples of non-isomorphisms of A_R.

2. The case $p = 0$. This case practically presents an Hadamard product of $y(z)$ with $b(z) = \sum_{k=0}^{\infty} b_k z^k$. First we present here the description from [5] of the commutant C_{M_p} of the operator M_p when it preserves the powers, i.e. when $p = 0$.

Theorem 1 (Hristova [5])**.** *A continuous linear operator $L : A_R \to A_R$ commutes with the operator M_0 defined by*

$$M_0 y(z) = M_0 \left(\sum_{k=0}^{\infty} a_k z^k \right) = \sum_{k=0}^{\infty} a_k b_k z^k, \tag{4}$$

where $b_k \neq 0$ and $b_k \neq b_j$ for $k \neq j$, if and only if it has the form

$$Ly(z) = \sum_{k=0}^{\infty} a_k d_k z^k, \quad d_k \neq 0, \tag{5}$$

provided $L : A_R \to A_R$.

Now we give some natural sufficient conditions for the operators L in order an operator $L \in C_{M_p}$ to be an automorphism of A_R.

Theorem 2. *Let a continuous linear operator $L : A_R \to A_R$ commute with the operator M_0 defined by (4) and the complex sequence $\{d_k\}_{k=0}^{\infty}$ is such that $d_k \neq 0, k = 0, 1, 2, \ldots$, and $\liminf_{k\to\infty} \sqrt[k]{|d_k|} \geq 1$, then L is an automorphism of A_R.*

Proof. The injectivity of the operator $L : A_R \to A_R$ is obvious since the condition $d_k \neq 0$, $k = 0, 1, 2, \ldots$, ensures that the equation

$$Ly(z) = \sum_{k=0}^{\infty} a_k d_k z^k \equiv 0 \tag{6}$$

has only the trivial solution $y(z) \equiv 0$.

Let us prove the surjectivity. Take an arbitrary analytic function

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \in A_R.$$

We are to find a $y(z) \in A_R$ such that $Ly(z) = f(z)$, i.e.

$$\sum_{k=0}^{\infty} a_k d_k z^k = \sum_{k=0}^{\infty} c_k z^k.$$

Comparing the coefficients, we get

$$a_k = \frac{c_k}{d_k}, \quad k = 0, 1, 2, \ldots. \tag{7}$$

The function $f(z)$ has a radius of convergence of its power expansion at least R. This means

$$\limsup_{k\to\infty} \sqrt[k]{|c_k|} \leq \frac{1}{R}.$$

Then the condition $\liminf\limits_{k\to\infty} \sqrt[k]{|d_k|} \geq 1$ and (7) imply

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k]{\left|\frac{c_k}{d_k}\right|} = \limsup_{k\to\infty} \frac{\sqrt[k]{|c_k|}}{\sqrt[k]{|d_k|}} = \frac{\limsup\limits_{k\to\infty} \sqrt[k]{|c_k|}}{\liminf\limits_{k\to\infty} \sqrt[k]{|d_k|}} \leq \frac{1}{R}.$$

This means that f is an image of a function $y \in A_R$. $\square$

Note: The injectivity does not hold for example if $d_k = 0$ for at least one k, because then the coefficient a_k of z^k in the power expansion of y can be taken to be nonzero. Without the additional condition $\liminf\limits_{k\to\infty} \sqrt[k]{|d_k|} \geq 1$ the surjectivity can also fail for some choices of the sequence $\{d_k\}_{k=0}^{\infty}$.

3. The case $p = 1$. The commutant C_{M_p} of the general operator M_p given by (1), (2) or (3), was described in [5] for $p \geq 1$, but for our purposes here we give only the result for $p = 1$ in a slightly different form of the representation in which the equal powers of z are gathered at one place.

Theorem 3 (Hristova [5])**.** *A continuous linear operator $L : A_R \to A_R$ commutes with the operator M_p defined by* (1), (2) *or* (3), *with $p = 1$ if and only if it has the form*

$$Ly(z) = \sum_{m=0}^{\infty} \left(a_0 c_{0,m} + \sum_{k=1}^{m} a_k \cdot \frac{b_{m-1}\dots b_{m-k}}{b_{k-1}\dots b_0}\, c_{0,m-k} \right) z^m, \tag{8}$$

where $a_k = \left[\dfrac{y^{(k)}(z)}{k!}\right]_{z=0}$ and the complex coefficients $c_{0,m}$ can be chosen arbitrarily for $m = 0, 1, 2, \dots,$ provided the series in the right-hand side has a radius of convergence at least R, i.e. $L : A_R \to A_R$.

Proposition 1. *If $p = 1$ and L is an operator of the commutant C_{M_1} with $(L(z^0))(0) \neq 0$, then it is an injective mapping.*

Proof. In order to prove the injectivity of the operator L, it is enough to show that its kernel is trivial, i.e. that $Ly(z) \equiv 0$ implies $y = \sum\limits_{k=0}^{\infty} a_k z^k \equiv 0$, which means that if all brackets in (8) vanish, then $a_k = 0$ for $k = 0, 1, 2, \dots$. Let us write the first equations of the corresponding infinite system for the coefficients a_k:

$$\begin{aligned} a_0 c_{0,0} &= 0 \\ a_0 c_{0,1} + a_1 c_{0,0} &= 0 \\ a_0 c_{0,2} + a_1 \frac{b_1}{b_0} c_{0,1} + a_2 c_{0,0} &= 0 \\ \dots &= 0\,. \end{aligned} \tag{9}$$

Note that the condition $(L(1))(0) \neq 0$ means $c_{0,0} \neq 0$ since if $1 \equiv f(z) = \sum\limits_{j=0}^{\infty} \alpha_j z^j$, then $\alpha_0 = 1$ and $\alpha_j = 0$ for $j = 1, 2, \dots$.

Now $a_0 = 0$ from the first equation of the system. Substituting in the next equation we get $a_1 = 0$. Continuing in the same way, finally, $a_k = 0$ for all $k = 0, 1, 2, \dots$ and the injectivity is proved. □

Next we give a sufficient condition for an operator of the commutant C_{M_1} to be a surjective mapping:

Proposition 2. *Let the operator M_p, given by* (1), (3) *or* (2), *be such that $p = 1$ and the sequence $\{b_k\}_{k=1}^{\infty}$ is bounded. If L is an operator of the commutant C_{M_1} described in* (8) *with $(L(z^0))(z) = c_{0,0} + c_{0,1}z$, i.e. $c_{0,m} = 0$ for $m \geq 2$, then $L : A_R \to A_R$ is a surjective mapping.*

Proof. We have to show that if $L \in C_{M_1}$ and $f(z) = \sum_{m=0}^{\infty} d_m z^m \in A_R$, then it is possible to find a function $y(z) = \sum_{n=0}^{\infty} a_n z^n \in A_R$ with $Ly(z) = f(z)$.

Using (8) and comparing the coefficients of the equal powers we have to solve a system with the same left-hand sides as (9) but with the coefficients of f as right-hand sides

$$\begin{aligned} a_0 c_{0,0} &= d_0 \\ a_0 c_{0,1} + a_1 c_{0,0} &= d_1 \\ &\dots \\ a_0 c_{0,m} + \dots + a_s \frac{b_{m-1} \dots b_{m-s}}{b_{s-1} \dots b_0} c_{0,m-s} + \dots + a_m c_{0,0} &= d_m. \\ &\dots \end{aligned} \tag{10}$$

The conditions $(L(z^0))(z) = c_{0,0} + c_{0,1}z$ and $c_{0,m} = 0$ for $m \geq 2$ reduce the left-hand sides of all equations in (10) to only two terms each

$$\begin{aligned} a_0 c_{0,0} &= d_0 \\ a_0 c_{0,1} + a_1 c_{0,0} &= d_1 \\ &\dots \\ a_{m-1} \frac{b_{m-1}}{b_0} c_{0,1} + a_m c_{0,0} &= d_m \\ &\dots \end{aligned} \tag{11}$$

and then the solution is

$$\begin{aligned} a_0 &= \frac{1}{c_{0,0}} d_0 \\ a_1 &= \frac{1}{c_{0,0}} \left(d_1 - d_0 \frac{c_{0,1}}{c_{0,0}} \right) \\ &\dots \\ a_m &= \frac{1}{c_{0,0}} \left(d_m - d_{m-1} \frac{b_{m-1}}{b_0} \left(\frac{c_{0,1}}{c_{0,0}} \right) + \dots + (-1)^m d_0 \frac{b_{m-1} \dots b_0}{b_0^m} \left(\frac{c_{0,1}}{c_{0,0}} \right)^m \right) \\ &\dots \end{aligned} \tag{12}$$

Let us suppose that the series representing $f(z)$ has as a radius of convergence R, i.e.

$$\limsup_{m \to \infty} \sqrt[m]{|d_m|} = \frac{1}{R}. \tag{13}$$

We have to make an upper estimate of $|a_m|$, which ensures at least the same radius of convergence of the series representing the solution $y(z)$ of $Ly(z) = f(z)$, i.e.

$$\limsup_{m\to\infty} \sqrt[m]{|a_m|} \le \frac{1}{R}.$$

Let us use the condition that the coefficients b_k in the definition (1) of the operator M_1 are bounded by a constant B

$$|b_k| < B, \quad k = 0, 1, 2, \dots. \tag{14}$$

Let us also denote

$$q = \left|\frac{Bc_{0,1}}{b_0 c_{0,0}}\right|. \tag{15}$$

Now take an arbitrary R_1 with $0 < R_1 < R$ and $0 < qR_1 < 1$. Then (13) implies that there exists some k_0, such that

$$|d_k| < \frac{1}{R_1^k}, \qquad \text{for } k > k_0. \tag{16}$$

For $m > k_0$ we have to separate the sum in the bracket of (12) into two sums, the first for $m \ge s \ge k_0 + 1$ and the second for $k_0 \ge s \ge 0$, in order to estimate them differently. For the first sum

$$\begin{aligned} &\left| d_m - d_{m-1}\frac{b_{m-1}}{b_0}\left(\frac{c_{0,1}}{c_{0,0}}\right) + \dots + (-1)^{m-k_0+1} d_{k_0+1}\frac{b_{m-1}\dots b_{k_0+1}}{b_0^{m-k_0-1}}\left(\frac{c_{0,1}}{c_{0,0}}\right)^{m-k_0-1} \right| \\ &\qquad \le |d_m| + |d_{m-1}|q + \dots + |d_{k_0+1}|q^{m-k_0-1} \\ &\qquad \le \left(\frac{1}{R_1^m} + \frac{q}{R_1^{m-1}} + \dots + \frac{q^{m-k_0-1}}{R_1^{k_0+1}}\right). \end{aligned} \tag{17}$$

In order to estimate the second sum we artificially multiply the terms by suitable quotients $\dfrac{R_1^l}{R_1^l} = 1$, $k_0 \ge l \ge 0$, and put $D = \max\limits_{0\le s\le k_0} |R_1^s d_{m-s}|$:

$$\begin{aligned} &\left| (-1)^{m-k_0}\frac{R_1^{k_0}}{R_1^{k_0}} d_{k_0}\frac{b_{m-1}\dots b_{k_0}}{b_0^{m-k_0}}\left(\frac{c_{0,1}}{c_{0,0}}\right)^{m-k_0} + \dots + (-1)^m d_0 \frac{b_{m-1}\dots b_0}{b_0^m}\left(\frac{c_{0,1}}{c_{0,0}}\right)^m \right| \\ &\qquad \le D\left(\frac{1}{R_1^{k_0}}q^{m-k_0} + \frac{1}{R_1^{k_0-1}}q^{m-k_0+1} + \dots + q^m\right). \end{aligned} \tag{18}$$

If we want to combine (17) and (18), take a new coefficient $H = \max\left\{\dfrac{1}{R_1}, D\right\}$. Then

$$|a_m| \le \frac{H}{c_{0,0}R_1^m}\left(1 + qR_1 + \dots + (qR_1)^m\right) \le \frac{1}{R_1^m}\cdot\frac{H}{c_{0,0}(1-qR_1)}\ , \tag{19}$$

using the formula for the sum of an infinite geometric progression. Finally, the radius of convergence of the series expansion of y is at least R_1 since

$$(20)\qquad \limsup_{m\to\infty} \sqrt[m]{|a_m|} \le \frac{1}{R_1} \limsup_{m\to\infty} \sqrt[m]{\frac{H}{c_{0,0}(1-qR_1)}} = \frac{1}{R_1} \le \frac{1}{R}\,.$$

Thus we were able to find a class of operators of the commutant C_{M_1} which are surjective mappings. □

Combining the results in Proposition 1 and Proposition 2 we can state the following

Theorem 4. *Let the operator M_p, given by* (1), (2) *or* (3), *be such that $p = 1$ and the sequence $\{b_k\}_{k=1}^{\infty}$ be bounded. If L is an operator of the commutant C_{M_1} described in* (8) *with $(L(z^0))(z) = c_{0,0} + c_{0,1}z$, i.e. $c_{0,m} = 0$ for $m \ge 2$, then $L : A_R \to A_R$ is an automorphism.*

4. The case $p \ge 2$. In the previous section we used the particular description (8) for $p = 1$ of the commutant C_{M_p} of the general operator M_p given by (1), (3) or (2). Here we give the general description from [5] for $p \ge 1$, but again modified with the equal powers of z gathered at one place.

Theorem 5 (Hristova [5]). *A continuous linear operator $L : A_R \to A_R$ commutes with the operator M_p defined by* (1), (2) *or* (3), *with $p \ge 1$ if and only if it has the form*

$$(21)\quad Ly(z) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{p-1} a_k c_{k,m} + \sum_{k=p}^{\left[\frac{m}{p}\right]p+p-1} a_k \cdot \frac{b_{m-p} \dots b_{m-\left[\frac{k}{p}\right]p}}{b_{k-p} \dots b_{k-\left[\frac{k}{p}\right]p}}\, c_{k-\left[\frac{k}{p}\right]p,\, m-\left[\frac{k}{p}\right]p} \right) z^m,$$

where $a_k = \left[\dfrac{y^{(k)}(z)}{k!}\right]_{z=0}$ and the complex coefficients $c_{k,m}$ can be chosen arbitrarily for $0 \le k \le p-1$ and $m = 0, 1, 2, \dots$, provided the series in the right-hand side has a radius of convergence at least R, i.e. $L : A_R \to A_R$.

This time we do not prove some general theorem as in Section 3. It seems that using these methods it is difficult to find necessary and sufficient conditions in the case $p \ge 2$ for the operators of C_{M_p} to be automorphisms. Therefore we will give only some examples without proofs for this case since the reasonings are similar to those made above, namely, solving infinite systems for the coefficients of some functions.

Example 1. *Let the operator M_p be given by* (1), (2) *or* (3), *and L be an operator of the commutant C_{M_p} given by* (21) *such that*

$$\begin{aligned} (L(z^0))(z) &= c_{0,0} + \dots + c_{0,p-1} z^{p-1}, \\ \dots &= \dots \\ (L(z^{p-1}))(z) &= c_{p-1,0} + \dots + c_{p-1,p-1} z^{p-1}, \end{aligned}$$

i.e. $c_{0,m} = \dots = c_{p-1,m} = 0$ for $m \ge p$. Then the condition

$$\begin{vmatrix} c_{0,0} & \cdots & c_{0,p-1} \\ \vdots & \ddots & \vdots \\ c_{p-1,0} & \cdots & c_{p-1,p-1} \end{vmatrix} = 0$$

implies that the operator L is not injective.

The reason is that one can find a nontrivial solution for the first p coefficients $(a_0, \ldots, a_{p-1}) \neq (0, \ldots, 0)$ of $y(z) = \sum_{k=0}^{\infty} a_k z^k$ from the infinite system representing the equation $Ly(z) \equiv 0$ and obtained by equating the coefficients of the powers of z to zero, i.e. the kernel of the operator L contains a nontrivial analytic function $y(z) \not\equiv 0$.

Example 2. *Let M_p be as in* Example 1*, but with $c_{q,q} \neq 0$ for $0 \leq q \leq p-1$, $c_{q,r} = 0$ for $q \neq r$, $0 \leq q \leq p-1$, $0 \leq r \leq p-1$, and with $c_{0,m} = \cdots = c_{p-1,m} = 0$ for $m \geq p$. Then the operator L is an automorphism.*

The proof will be omitted since the systems to be solved resemble the ones in Proposition 1 and Proposition 2 due to the condition that $c_{q,r} \neq 0$ only for $q = r$.

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