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NON-OSCILLATORY SOLUTIONS OF ODD ORDER LINEAR FUNCTIONAL DIFFERENTIAL SYSTEM OF NEUTRAL TYPE WITH DISTRIBUTED DELAY

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Abstract

In the present paper we establish explicit and easy computable sufficient conditions for existing of several types of non-oscillatory solutions of linear delayed system of neutral type with distributed delay. The results are proved by numerical range technique, and generally they are applicable in the case of non-monotone measures too.

Key words: neutral type, distributed delay, non-oscillatory solution, logarithmic norm, numerical range

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1. Introduction. The neutral delay differential equations have applications in physics, biology and other real world life problems. For example, the neutral equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits). The first detailed study of the linear delay differential equations and system with distributed delay (fundamental theory, stability, oscillation behaviour, etc.) was done by A. D. Myshkis in his fundamental monograph [1]. As an important part of the qualitative theory the oscillation theory of the functional differential equations has received serious attention – for more details see [2–6] and the references therein.

Our results extend and improve the results obtained in [7] in the particular case of one constant delay in the neutral part of the considered system, for the general case of distributed delay in the neutral part. In the present work we generalize some results in [8] proved for the case of monotonic and continuous functions, to the case when the same functions are only monotonic.

2. Preliminaries. We consider the following linear delayed system of neutral type with distributed delay

$$(1)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)+\delta\int_{-\tau}^{0}\mathrm{d}v(s)x(t+s)\right)-\int_{-\sigma}^{0}\mathrm{d}u(s)x(t+s)=0,$$

where $\delta\in\{-1,1\},\sigma>0,\tau>0,x:[0,\infty)\to R^n, v:[-\tau,0]\to RL^n, u:[-\sigma,0]\to RL^n$. By RL^n we denote the linear space of the $n\times n$ matrices $A=\{a_{ij}\}_{i,j=1}^n, a_{ij}\in R^1, n\geq 1$ is integer.

For each $a,b\in R^1, a<b$ we will denote by $BV[a,b]$ the linear space of matrix valued functions $v:[a,b]\to RL^n$ with bounded variation on $[a,b]$.

We suppose that for the functions $v:[-\tau,0]\to RL^n$ and $u:[-\sigma,0]\to RL^n$ the following conditions (S) are fulfilled:

(S1) The function $v\in BV[-\tau,0]$ is left-side continuous on $[-\tau,0]$, $\lim\limits_{t\to+0}[\mathrm{Var}_{s\in[-t,0]}(v(s))]=v(0)=0$ and $\det(v(-\tau+0)-v(-\tau-0))\neq 0$.

(S2) The function $u\in BV[-\sigma,0]$ is left-side continuous on $[-\sigma,0)$ and $\det(u(-\sigma+0)-u(-\sigma-0))\neq 0$.

Remark 1. *The condition $(S1)$ means that the function $v\in BV[-\tau,0]$ is atomic at $s=-\tau$ and non-atomic at $s=0$. For the function $u\in BV[-\sigma,0]$ the condition $(S2)$ means that it is atomic at $s=-\sigma$ and for $s=0$ the function can be atomic as well as non-atomic.*

Let $A=\{a_{ij}\}\in RL^n$ is an arbitrary matrix, and $x\in R^n$ is an arbitrary vector-column. We will denote with $A^T=\{a_{ij}\}$ the transposed matrix and with $x^T=(x_1,\dots x_n)$ the transposed vector of x. We introduce the notations

$$||x||_k=(\sum_{i=1}^n|x_i|^k)^{\frac{1}{k}},\ 1\leq k<\infty,\ ||x||_\infty=\max_{1\leq i\leq n}\{|x_i|\},$$

$$||A||_k=\sup_{x\in R^n,x\neq 0}\frac{||Ax||_k}{||x||_k},\ k=1,2,\dots,\infty.$$

Definition 1 ([9])**.** *The function $\mu_k:RL^n\to R, k=1,2,\dots,\infty$ defined with the equality*

$$\mu_k(A)=\lim_{\epsilon\to+0}\frac{||I+\epsilon A||_k-1}{\epsilon},\ A\in RL^n,\ k=1,2,\dots,\infty,$$

where $I\in RL^n$ is the unit matrix, is called logarithmic norm (Lozinskii measure).

Let us denote with $Sp(A)$ the spectrum of A and with $S(A):=\sup\{Re\lambda|\lambda\in Sp(A)\}$ the spectral bound of A. The logarithmic norm (Lozinskii measure) is not a norm (measure) in common sense, because it can take negative values, too. Below we present some basic properties of the logarithmic norm for square matrices.

Lemma 1 ([10, 11]). *Let $A, B \in RL^n,\ k = 1, 2, \ldots, \infty,\ \alpha \geq 0, \beta \in R^1$. Then the following relations hold*:

i. $\mu_k(\alpha A + \beta I) = \alpha\mu_k(A) + \beta;\ \mu_k(A) + \mu_k(-A) \geq 0$;

ii. $-||A||_k \leq -\mu_k(-A) \leq \mu_k(A) \leq ||A||_k;\ |\mu_k(A) - \mu_k(B)| \leq ||A - B||_k$;

iii. $\mu_k(-A) \leq Re\lambda \leq \mu_k(A)$ *for each* $\lambda \in Sp(A)$;

iv. $\mu_1(A) = \sup\limits_{1\leq j\leq n} \{a_{jj} + \sum\limits_{i=1, i\neq j}^{n} |a_{ij}|\}, \quad \mu_2(A) = \frac{1}{2}S(A + A^T)$,

$\mu_\infty(A) = \sup\limits_{1\leq i\leq n} \{a_{ii} + \sum\limits_{j=1, j\neq i}^{n} |a_{ij}|\}$.

Below, in general, we denote by $\mu(A)$ each of $\mu_k(A),\ k = 1, 2, \ldots, \infty$.

Remark 2. *It is clear that from Lemma 1 it follows that the logarithmic norm $\mu_k : LR^n \to R^1$ is a continuous function, if the topology in RL^n is induced by the norm $||.||_k,\ k = 1, 2, \ldots, \infty$. Moreover, if $v \in BV[a, b]$, then the function $\mu \circ v : [a, b] \to R^1$ has bounded variation on $[a, b]$ too.*

Let $t^* = \max(\tau, \sigma)$. We denote with $C([-t^*, 0], R^n)$ the space of the continuous functions $\varphi : [-t^*, 0] \to R^n$ and with $C^1([t_0, \infty), R^n), t_0 \in [-t^*, \infty)$ the space of the continuously differentiable functions $x : [t_0, \infty) \to R^n$.

Definition 2. *The function $x \in C([t^*, +\infty), R^n)$ is called a solution of system* (1) *if $x(t) + \delta_1 \int\limits_{-\tau}^{0} \mathrm{d}v(s)x(t + s) \in C^1([0, +\infty), R^n)$, $x(t)$ satisfies system* (1) *for each $t \geq 0$ and $x(t) = \varphi(t),\ t \in [-t^*, 0]$ for some initial function $\varphi \in C([-t^*, 0], R^n)$.*

Definition 3. *The solution $x(t) = (x_1(t), \ldots, x_n(t))^T$ of* (1) *is said to be oscillatory* (*strongly oscillatory*) *if there exist an index i, $1 \leq i \leq n$ and a sequence $\{t_k^i\}_{k=1}^{+\infty}$, $\lim\limits_{k\to+\infty} t_k^i = +\infty$ such that $x_i(t_k^i) = 0$ $(x_i(t_k^i)x_i(t_{k+1}^i) < 0)$ for each $k \geq 1$. If all solutions of system* (1) *are oscillatory, we call system* (1) *oscillatory.*

Definition 4. *The solution $x(t) = (x_1(t), \ldots, x_n(t))^T$ of* (1) *is said to be non-oscillatory, if for each index i, $1 \leq i \leq n$, there exists a point $t_i \geq 0$ such that $|x_i(t)| > 0$ for each $t \geq t_i$.*

Definition 5. *The function $w \in BV[a, b]$ will be called monotone increasing* (*decreasing*) *on $[a, b]$ with respect to the logarithmic norm μ , if for each number $c, d \in [a, b],\ c < d\ (c > d)$ the inequality $\mu(w(c) - w(d)) \leq 0$ is fulfilled. If this inequality is strict, then we call the function strictly increasing* (*decreasing*) *on $[a, b]$ with respect to the logarithmic norm μ.*

Let us define the function $F : R^1 \to RL^n$ by the following equality:

$$F(\lambda) = \lambda I + \lambda\delta \int\limits_{-\tau}^{0} e^{\lambda s}\mathrm{d}v(s) - \int\limits_{-\sigma}^{0} e^{\lambda s}\mathrm{d}u(s). \tag{2}$$

Obviously, the function $F(\lambda)$ is continuous for $\lambda \in R^1$ if the topology in

RL^n is induced by the norm $||.||_k$, $k = 1, 2, \ldots, +\infty$. Moreover, the function $\mu \circ F : R^1 \to R^1$ is continuous for $\lambda \in R^1$, too.

Lemma 2. *The necessary and sufficient condition for existing of a non-oscillatory solution of system* (1) *of the type* $x(t) = e^{\lambda t} C(\lambda)$, *where* $\lambda \in R^1$, $C(\lambda) \in R^n$, *is the characteristic equation*

$$\det\left(\lambda(I + \delta \int_{-\tau}^{0} e^{\lambda s} \mathrm{d}v(s)) - \int_{-\sigma}^{0} e^{\lambda s} \mathrm{d}u(s)\right) = 0 \tag{3}$$

to have at least one real root.

3. Main results. In this section are given explicit and easy computable sufficient conditions for the existing of several types of non-oscillatory solutions of (1). The results are proved for the case when the functions $u \in BV[-\sigma, 0]$ and $v \in BV[-\tau, 0]$ are non-monotonic with respect to the logarithmic norm as in [7].

For each function $w = \{w_{ij}\} \in BV[-t^*, 0]$, $n \geq 1$ we denote by $J_{w_{ij}}[a, b]$ the set points of jumps of the function $w_{ij} : [-t^*, 0] \to R^1$, i.e.

$$J_{w_{ij}}[a, b] = \{t \in [a, b] \mid |w_{ij}(t + 0) - w_{ij}(t - 0)| > 0\}.$$

By $J_w[a, b]$ we will denote the set $J_w[a, b] = \bigcup_{i,j=1}^{n} J_{w_{ij}}[a, b]$ and by $E_w[a, b]$ the set $E_w[a, b] = [a, b] \backslash J_w[a, b]$.

Lemma 3. *Let the following conditions be fulfilled*:

1. *The function* $w \in BV[-t^*, 0]$.

2. *The function* $f : [-t^*, 0] \to R^1$ *is monotonous and has a constant sign on* $[-t^*, 0]$.

3. $J_w[-t^*, 0] \bigcap J_f[-t^*, 0] = \varnothing$.

Then the following inequalities hold:

a) *If* f *is decreasing and positive* (*negative*), *then*

$$\mu(\int_{-t^*}^{0} f(s) \mathrm{d}w(s)) \leq \int_{-t^*}^{0} f(s) \mathrm{d}\mu(w(s) - w(-t^*));$$

$$\left(\mu(\int_{-t^*}^{0} f(s) \mathrm{d}w(s)) \leq \int_{-t^*}^{0} f(s) \mathrm{d}\mu(w(s) - w(0))\right).$$

b) *If* f *is increasing and positive* (*negative*), *then*

$$\mu(\int_{-t^*}^{0} f(s) \mathrm{d}w(s)) \leq -\int_{-t^*}^{0} f(s) \mathrm{d}\mu(w(0) - w(s));$$

$$\left(\mu(\int_{-t^*}^{0} f(s) \mathrm{d}w(s)) \leq -\int_{-t^*}^{0} f(s) \mathrm{d}\mu(w(-t^*) - w(s))\right).$$

Remark 3. *The assertions of Lemma* 3 *have been proved as Lemma* 2.2 *in* [8] *for the case of monotonic and continuous functions* $f(x)$. *Condition* 3 *of Lemma* 3, *introduced by us permits to generalize the assertion to the case when the function* $f(x)$ *is monotonic only. Since condition* 3 *of Lemma* 3 *is evidently fulfilled when the function* $f(x)$ *is monotonic and continuous, then this condition is not an additional restriction for this case.*

Remark 4. *It is easy to see that* $\lambda = 0$ *is a root of equation* (3) *if and only if, when* $\det(u(-\sigma) - u(0)) = 0$ *and then without any additional conditions, system* (1) *has a bounded non-oscillatory solution. Then from condition* $\det(u(-\sigma) - u(0)) \neq 0$ *used below, it follows only that* $\lambda = 0$ *is not a root of the equation* (3).

Theorem 1. *Let the following conditions be fulfilled*:

1. *Conditions* (*S*) *hold,* $\delta = -1$ ($\delta = +1$) *and the number* n *is odd.*
2. $\det(u(-\sigma) - u(0)) < 0$.
3. $\sup\limits_{s \in E_v[-\tau,0]} \mu(-v(s)) < 1$ ($\sup\limits_{s \in E_v[-\tau,0]} \mu(v(s)) < 1$).

Then system (1) *has at least one unbounded non-oscillatory solution.*

Proof. Let $\delta = -1$ and $\sup\limits_{s \in E_v[-\tau,0]} \mu(-v(s)) < 1$. Since the function $\det F(\lambda)$ is continuous for $\lambda \in R^1$, it is enough to prove that there exist two numbers $\lambda_1, \lambda_2 \in [0, +\infty)$, $\lambda_1 \neq \lambda_2$, such that $\det(F(\lambda_1))\det(F(\lambda_2)) \leq 0$. Taking into account that n is an odd number, according to Lemma 2 and Gerschgorin's theorem this inequality will be true, if for example $\mu(F(\lambda_1)) \leq 0$ and $\mu(-F(\lambda_2)) \leq 0$. From condition 2 of the theorem it follows that $\det(F(0)) < 0$ and thus we can choose $\lambda_1 = 0$. Since $F(\lambda)$ is continuous and if we suppose that $\det(F(\lambda)) \neq 0$ for $\lambda \in [0, +\infty)$, then $\det(F(\lambda)) < 0$ for $\lambda \in [0, +\infty)$. Condition 1 of the theorem implies that the equation $\det(\xi I - F(\lambda)) = 0$ for each $\lambda \geq 0$ has at least one real negative root $\xi(\lambda)$, i.e. the matrix $F(\lambda)$ has at least one real negative eigenvalue $\xi(\lambda)$. Then from Lemma 1 it follows that $\mu(-F(\lambda)) > 0$ for $\lambda \in [0, +\infty)$.

From (2), Lemma 1 and Lemma 3 we receive the following estimation:

$$\mu(-F(\lambda)) \leq -\lambda - \lambda \int_{-\tau}^{0} e^{\lambda s} \mathrm{d}\mu(v(0) - v(s)) - \int_{-\sigma}^{0} e^{\lambda s} \mathrm{d}\mu(u(0) - u(s)). \tag{4}$$

Since $v \in BV[-\tau, 0]$ and therefore it has countably many bounded jumps, then for the second addend on the right-side of (4) after integration of parts, we receive the estimation

$$\begin{aligned} -\lambda \int_{-\tau}^{0} e^{\lambda s} \mathrm{d}\mu(v(0) - v(s)) &\leq \lambda e^{-\lambda\tau} \mu(v(0) - v(-\tau)) \\ &- \lambda e^{-\lambda\tau} \sup_{s \in E_v[-\tau,0]} \mu(v(0) - v(s)) + \lambda \sup_{s \in E_v[-\tau,0]} \mu(v(0) - v(s)). \end{aligned} \tag{5}$$

For the third addend on the right-side of (4) in a similar way, we obtain

$$
(6)\qquad -\int_{-\sigma}^{0} e^{\lambda s}\mathrm{d}\mu(u(0)-u(s)) \le e^{-\lambda\sigma}\mu(u(0)-u(-\sigma)) - e^{-\lambda\sigma}\sup_{s\in[-\sigma,0]}\mu(u(0)-u(s)) + \sup_{s\in[-\sigma,0]}\mu(u(0)-u(s)).
$$

Since $\sup\limits_{s\in E_v[-\tau,0]}\mu(-v(s)) < 1$, then taking into account (4), (5) and (6), we can conclude that $\mu(-F(\lambda)) \le 0$ if $\lambda > 0$ is sufficiently large, which is a contradiction. Therefore, from Lemma 2 it follows that system (1) has at least one unbounded non-oscillatory solution.

The other case can be proved similarly. □

Corollary 1. *Let the following conditions be fulfilled*:

1. *Conditions* (S) *hold and the number* n *is odd.*

2. $\mu(u(-\sigma)-u(0)) \le 0$ *and* $\det(u(-\sigma)-u(0)) \ne 0$.

3. *One of the following two conditions holds*:

3.1. $\delta = -1$ *and* $\sup\limits_{s\in E_v[-\tau,0]}\mu(-v(s)) < 1$.

3.2. $\delta = 1$ *and* $\sup\limits_{s\in E_v[-\tau,0]}\mu(v(s)) < 1$.

Then system (1) *has at least one unbounded non-oscillatory solution.*

Remark 5. *In* [7] *the results from Theorems* 1 *and* 2 *are obtained in the particular case when* $\delta = 1$ *and* $v(s) = H(-(s+\tau))A$, $A \in RL^n$, *where* $H(s)$ *is the Heaviside function, under two additional restrictive conditions – the function* $u(s)$ *must be nonatomic at zero and monotonic according the logarithmic norm. Moreover, the introduced by us conditions* 3 *of Theorems* 1 *and* 2 *are evidently fulfilled in the case considered in* [7], *i.e. these conditions are not additional restrictions for this case.*

Remark 6. *Generally speaking from the condition* $\mu(u(-\sigma)-u(0)) < 0$ *it does not follow that the function* $u(s)$ *is monotonous on* $[-\sigma,0]$ *with respect to the logarithmic norm.*

For some applications it is important to know how the relation between τ and σ influences on the existence of non-oscillatory solutions for system (1) and their asymptotical behaviour.

Theorem 2. *Let the following conditions be fulfilled*:

1. *Conditions* (S) *hold and the number* n *is odd.*

2. $\det(u(-\sigma)-u(0)) < 0$.

3. *One of the following two conditions holds*:

3.1. $\delta = -1$ *and* $\sup\limits_{s\in E_v[-\tau,0]}\mu(v(-\tau)-v(s)) < 0$.

3.2. $\delta = 1$ *and* $\sup\limits_{s\in E_v[-\tau,0]}\mu(v(s)-v(-\tau)) < 0$.

4. $\tau > \sigma$.

Then system (1) *has at least one bounded non-oscillatory solution* $x(t) = (x_1(t), \dots, x_n(t))^T$ *such that* $\lim\limits_{t\to+\infty} x_i(t) = 0$ *for each* i, $1 \le i \le n$.

Example 1. Let in system (1) $\delta = -1$ and $\tau > 0$. Consider the function $v(s)$ in the form $v(s) = \sum_{k=1}^{m} H_k(-(s+\tau_k))A_k$, where $A_k \in RL^n, \tau_k \in (0,\tau], k = 1,2,\dots,m,\ 0 < \tau_1 < \tau_2 < \dots < \tau_m = \tau$. Let the relation $\sup\limits_k \mu(\sum\limits_{i=k}^{m}(A_i)) < 0$, $k = 1,2,\dots,m$ hold. Since $E_v[-\tau,0] = (-\tau_1,0]\bigcup(\bigcup\limits_{i=1}^{m-1}(-\tau_{i+1},-\tau_i))$, then by means of Lemma 1, it is not difficult to see that for the function $v(s)$ the condition 3.1 of Theorem 2 holds. Obviously, this relation is fulfilled if $\mu(A_k) \le 0,\ k = 1,2,\dots,m$.

4. Discussion. Since the matrices $A \in RL^n$ when $n = 1$ are real numbers, then $\mu(A) = \det A$. Then the two cases mentioned in conditions 3 of Theorem 1 are identical. Since for $A \in RL^n,\ n \ge 3$ generally speaking $-\mu(A) \ne \mu(-A)$, then we can see that the two cases mentioned in conditions 3 of Theorem 1 are different. The appearance of these cases is the effect of the high dimension.

Example 2. Let us consider system (1), when $n = 3,\ \delta = 1, A, B_0, B_1 \in RL^3$ and let consider the functions $v(s), u(s)$ in the form

$$v(s) = H(-(s+2))A,\ u(s) = H(-(s+3))B_1 + H(s)B_0.$$

If we choose the matrices as follows:

$$A = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ B_0 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},\ B_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

then all conditions of Theorem 1 are fulfilled, and the characteristic equation (3) will have the form $\det(\lambda(I - Ae^{-2\lambda}) - (B_0 - B_1e^{-3\lambda})) = 0$. For example, calculating with Wolfram Mathematica, we receive that this equation has a real positive root $s = 4.9998$. Thus system (1) has at least one unbounded non-oscillatory solution.

Remark 7. *The Example* 2 *is essentially distinguished from the example introduced in* [7] *because the additional requirement for* $u(s)$ *to be nonatomic at zero implies that the matrix* B_0 *in our example must be always zero. Thus our results are new even in the cases considered* [7].

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