Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 66, No 6, 2013

MATHEMATIQUES
Méthodes de calculs numériques

CONVERGENCE OF THE WEIERSTRASS METHOD FOR SIMULTANEOUS APPROXIMATION OF POLYNOMIAL ZEROS

Petko D. Proinov, Milena D. Petkova

(*Submitted by Academician Bl. Sendov on January 1, 2013*)

Abstract

In 1891, Weierstrass presented his famous iterative method for finding all the zeros of a polynomial simultaneously. In this paper we establish three new local convergence theorems for the Weierstrass method with a posteriori and a priori error estimates. The main result of the paper generalizes, improves and complements some well known results of Dochev (1962), Kjurkchiev and Markov (1983) and Yakoubsohn (2002). The results are given for polynomials over an arbitrary normed field.

Key words: polynomial zeros, simultaneous methods, Weierstrass method, local convergence, error estimates

2010 Mathematics Subject Classification: 65H04, 12Y05

1. Introduction. Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes an arbitrary normed field, $\mathbb{K}[z]$ denotes the ring of polynomials (in one variable) over $\mathbb{K}$, and the vector space $\mathbb{K}^n$ is equipped with the p-norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for some $1 \le p \le \infty$. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. A vector $\xi \in \mathbb{K}^n$ is said to be a *root vector* of f if $f(z) = a_0 \prod_{i=1}^n (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$. Obviously, f has a root vector in $\mathbb{K}^n$ if and only if f splits over $\mathbb{K}$. If f has n simple zeros in $\mathbb{K}$, then $\operatorname{sep}(f)$ denotes the *separation number* of f which is defined to be the minimum distance between two zeros of f.

There are a lot of iterative methods for finding all zeros of f simultaneously (see, e.g., the monographs of Sendov, Andreev and Kjurkchiev [1], Kyurkchiev [2] and Petković [3] and references given therein). In 1891, Weierstrass [4] published his famous iterative method for simultaneous computation of all zeros of f. The *Weierstrass method* (known also as the Durand–Dochev–Kerner–Prešić method) is defined by the following iteration:

$$x^{k+1} = x^k - W(x^k), \qquad k = 0, 1, 2, \dots, \tag{1}$$

where the initial guess x^0 is a vector in $\mathbb{K}^n$ with distinct components and the operator W is defined in $\mathbb{K}^n$ by $W(x) = (W_1(x), \ldots, W_n(x))$ with

$$W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)} \qquad (i = 1, \ldots, n), \tag{2}$$

where a_0 is the leading coefficient of f. In 1962, Dochev [5] proved the following local convergence theorem for Weierstrass method.

Theorem 1.1 (Dochev [5])**.** *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with simple zeros, $\xi \in \mathbb{C}^n$ a root vector of f and $0 < h < 1$. Suppose $x^0 \in \mathbb{C}^n$ is an initial guess satisfying*

$$\|x^0 - \xi\|_\infty \leq \rho = \frac{(1+h)^{1/(n-1)} - 1}{2(1+h)^{1/(n-1)} - 1} \operatorname{sep}(f).$$

Then the Weierstrass iteration (1) *is well defined and converges quadratically to ξ with error estimate*

$$\|x^k - \xi\|_\infty \leq \rho\, h^{2^k - 1} \quad \textit{for all} \;\; k \geq 0.$$

The following theorem is an immediate consequence of Theorem 1.1.

Theorem 1.2. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with simple zeros, and let $\xi \in \mathbb{C}^n$ be a root vector of f. If the initial guess x^0 is such that*

$$\|x^0 - \xi\|_\infty < \frac{\sqrt[n-1]{2} - 1}{2\,\sqrt[n-1]{2} - 1} \operatorname{sep}(f),$$

then the Weierstrass iteration (1) *is well defined and converges quadratically to ξ.*

Theorem 1.2 was rediscovered by Wang and Zhao [6] in 1991 and by Niell [7] in 2001. In 1994, Hopkins et al. [8] have obtained a weaker version of Theorem 1.2. They have proved the quadratic convergence of Weierstrass iteration (1) under the condition

$$\|x^0 - \xi\|_\infty \leq \frac{\sqrt[n-1]{4} - 1}{2\,\sqrt[n-1]{4} + 1} \operatorname{sep}(f).$$

In 1983, Kjurkchiev and Markov [9] established the following convergence theorem for the Weierstrass method. Kjurkchiev and Markov's theorem and its proof can also be found in Sendov, Andreev and Kjurkchiev ([1], Section 18), Kyurkchiev ([2], Section 1.2) and Sendov and Popov ([10], Section 4.3.6).

Theorem 1.3 (Kjurkchiev–Markov [9])**.** *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with simple zeros, $\xi \in \mathbb{C}^n$ a root vector of f, $0 < h < 1$ and $0 < c \leq \operatorname{sep}(f)/(\alpha\, n + 1)$, where $\alpha = 1.763\ldots$ is the unique root of the equation $t = e^{1/t}$. If the initial guess $x^0 \in \mathbb{C}^n$ satisfies the inequality*

$$\|x^0 - \xi\|_\infty \leq c\, h,$$

then the Weierstrass iteration (1) *is well defined and converges to ξ with error estimate*

$$\|x^k - \xi\|_\infty \le c\, h^{2^k} \quad \textit{for all} \;\; k \ge 0.$$

In 2002, Yakoubsohn [11] published a γ-theorem for the Weierstrass method. He introduced the quantity

$$\gamma(f) = \max_{1 \le i \le n} \gamma(f, \xi_i), \quad \text{where} \quad \gamma(f, x) = \max_{k > 1} \left| \frac{f^{(k)}(x)}{k!\, f'(x)} \right|^{1/(k-1)}. \tag{3}$$

Recall that $\gamma(f, x)$ has been introduced by Smale in his famous work [12].

Theorem 1.4 (Yakoubsohn [11]). *Let $f \in \mathbb{C}[x]$ be a polynomial of degree $n \ge 2$ with simple zeros, and let ξ be a root-vector of f. Suppose x^0 is an initial guess in $\mathbb{C}^n$ such that*

$$\|x^0 - \xi\|_\infty \le \frac{1}{(5n-2)\, \gamma(f)}\,. \tag{4}$$

Then the Weierstrass iteration (1) *is well defined and converges quadratically to ξ with error estimate*

$$\|x^k - \xi\|_\infty \le \left(\frac{1}{2}\right)^{2^k - 1} \|x^0 - \xi\|_\infty \quad \textit{for all} \;\; k \ge 0. \tag{5}$$

In this paper we present three new local convergence theorems for the Weierstrass method with a posteriori and a priori error estimates. The main result of the paper (Theorem 2.1) generalizes, improves and complements the above mentioned results of Dochev [5], Kjurkchiev and Markov [9] and Yakoubsohn [11].

2. Main results. Throughout this section we use the following notations and conventions. We define the operator $d\colon \mathbb{K}^n \to \mathbb{R}^n$ by $d(x) = (d_1(x), \dots, d_n(x))$ with

$$d_i(x) = \min_{j \ne i} |x_i - x_j| \qquad (i = 1, 2, \dots, n).$$

In the sequel, we consider $\mathbb{K}^n$ as an algebra over $\mathbb{K}$ with multiplication defined by $xy = (x_1 y_1, \cdots, x_n y_n)$ for all $x, y \in \mathbb{C}^n$. Then we can consider the division

$$\frac{x}{y} = \left(\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}\right)$$

of two vectors $x, y \in \mathbb{K}^n$ if and only if $y_i \ne 0$ for all $i = 1, 2, \dots, n$. Throughout, we assume for convenient that $0^0 = 1$. Further, for a number p such that $1 \le p \le \infty$ we denote by q the conjugate exponent of p, i.e. q is defined by means of

$$1 \le q \le \infty \quad \text{and} \quad 1/p + 1/q = 1.$$

In 2009, Proinov [13] presented a general local convergence theory for iterative processes of the type

$$x_{k+1} = Tx_k, \qquad k = 0, 1, 2, \ldots, \tag{6}$$

where $T\colon D \subset X \to X$ is an iteration function in a metric space X. A central role in this theory plays the notion of a *function of initial conditions* of T (see Definition 3 below). In this theory, the convergence of an iterative process of type (6) is always studied with respect to a function of initial conditions.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$, and let $\xi \in \mathbb{K}^n$ be a root vector of f. In this paper we study the local convergence of the Weierstrass iteration (1) with respect to the function of initial conditions $E\colon \mathbb{K}^n \to \mathbb{R}_+$ defined as follows:

$$E(x) = E_f(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \qquad (1 \leq p \leq \infty). \tag{7}$$

Recently this function of initial conditions was used by Proinov and Ivanov [14] and Cholakov [15] for studying some other simultaneous iterative methods.

Now we are ready to state the main result of this paper which generalizes, improves and complements the above mentioned results of Dochev [5], Kjurkchiev and Markov [9] and Yakoubsohn [11].

Theorem 2.1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, ξ a root-vector of f, $1 \leq p \leq \infty$. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess satisfying*

$$E(x^0) = \left\| \frac{x^0 - \xi}{d(\xi)} \right\|_p < R(n, p) = \frac{2^{1/(n-1)} - 1}{2^{1/q}\left(2^{1/(n-1)} - 1\right) + (n-1)^{-1/p}}. \tag{8}$$

Then the Weierstrass iteration (1) *is well defined and converges quadratically to ξ with error estimates*

$$\|x^{k+1} - \xi\|_p \leq \lambda^{2^k} \|x^k - \xi\|_p \quad \textit{and} \quad \|x^k - \xi\|_p \leq \lambda^{2^k - 1} \|x^0 - \xi\|_p \quad \textit{for all} \quad k \geq 0, \tag{9}$$

where $\lambda = \phi(E(x^0))$ and the real function ϕ is defined by

$$\phi(t) = \left(1 + \frac{t}{(n-1)^{1/p}(1 - 2^{1/q}\, t)}\right)^{n-1} - 1. \tag{10}$$

Let us note that $R(n, p)$ is the unique root of the equation $\phi(t) = 1$ in the interval $(0, 1/2^{1/q})$. Taking into account that the function ϕ is continuous and strictly increasing on the interval $J = [0, R(n, p)]$ and that $\phi(J) = [0, 1]$ we can reformulate Theorem 2.1 in the following equivalent form.

Theorem 2.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, ξ a root-vector of f, $1 \leq p \leq \infty$ and $0 \leq h < 1$. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess such that*

$$\left\| \frac{x^0 - \xi}{d(\xi)} \right\|_p \leq R(n, p, h) = \frac{(1 + h)^{1/(n-1)} - 1}{2^{1/q}\left((1 + h)^{1/(n-1)} - 1\right) + (n-1)^{-1/p}}. \tag{11}$$

Then the Weierstrass iteration (1) *is well defined and converges quadratically to* ξ *with error estimates*

$$\|x^{k+1}-\xi\|_p \le h^{2^k}\, \|x^k-\xi\|_p \text{ and } \|x^k-\xi\|_p \le h^{2^k-1}\, \|x^0-\xi\|_p \text{ for all } k \ge 0. \tag{12}$$

It is easily seen that Theorem 1.1 is an immediate consequence of Theorem 2.2 and the obvious inequality

$$\left\| \frac{x-\xi}{d(\xi)} \right\|_p \le \frac{\| x-\xi\|_p}{\operatorname{sep}(f)} \tag{13}$$

which holds for every $x \in \mathbb{K}^n$ and $1 \le p \le \infty$.

Now we shall show that Theorem 2.2 is an improvement of Yakoubsohn's theorem [11] mentioned above. Suppose $x^0 \in \mathbb{K}^n$ satisfies Yakoubsohn's initial condition (4). We shall prove that x^0 satisfies the initial condition (11) with $p = \infty$ and $h = 1/2$. In other words, we have to show that

$$\left\| \frac{x^0-\xi}{d(\xi)} \right\|_\infty \le R(n, \infty, 1/2). \tag{14}$$

To prove this we need the following inequality:

$$\gamma(f) \ge \frac{1}{2\operatorname{sep}(f)} \tag{15}$$

which is due to Yakoubsohn [16]. From the inequalities (4), (15) and (13), we obtain

$$\left\| \frac{x^0-\xi}{d(\xi)} \right\|_\infty \le \frac{2}{5\,n-2}\,. \tag{16}$$

Using the well known inequality $e^t - 1 \ge t$, it is easy to prove the following estimate:

$$R(n, \infty, 1/2) \ge \frac{1}{A(n-1)+2}\,, \tag{17}$$

where $A = 1/\ln(3/2)$. If $n \ge 16$, then $2/(5n-2) \le 1/(An+2)$. If $2 \le n \le 15$, one can verify that $2/(5\,n-2) \le R(n, \infty, 1/2)$. In both cases, it follows from (16) and (17) that x^0 satisfies (14). Then from Theorem 2.2, we conclude that the Weierstrass iteration (1) is well defined and converges quadratically to ξ with error estimates

$$\|x^{k+1}-\xi\|_\infty \le \left(\frac{1}{2}\right)^{2^k} \|x^k-\xi\|_\infty \text{ and } \|x^k-\xi\|_\infty \le \left(\frac{1}{2}\right)^{2^k-1} \|x^0-\xi\|_\infty \text{ for all } k \ge 0.$$

Therefore, Theorem 1.4 is a consequence of Theorem 2.2.

It is easy to show that the function ϕ defined by (10) satisfies the inequality $\phi(t) \le t/R(n,p)$ for all $t \in [0, 1/2^{1/q})$. This yields the following simplified version of Theorem 2.1.

Theorem 2.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, ξ a root-vector of f and $1 \leq p \leq \infty$. Suppose that $x^0 \in \mathbb{C}^n$ is an initial guess satisfying* (8). *Then the Weierstrass iteration* (1) *is well defined and converges quadratically to ξ with error estimates*

$$(18)\quad \|x^{k+1} - \xi\|_p \leq \lambda^{2^k} \|x^k - \xi\|_p \text{ and } \|x^k - \xi\|_p \leq \lambda^{2^k - 1} \|x^0 - \xi\|_p \text{ for all } k \geq 0,$$

where $\lambda = E(x^0)/R(n,p)$.

Consider the quantity $R(n,p)$ defined in Theorem 2.1. It is easy to show the following estimate:

$$R(n,\infty) \geq \frac{1}{B(n-1)+2}\,, \tag{19}$$

where $B = 1/\ln 2$. It follows from (13) and (19) that Theorem 2.3 is an improvement of Kjurkchiev–Markov's result [9] mentioned above.

Remark 2.1. It follows from Section 4 that the Weierstrass iteration satisfies the following estimates under the initial condition of Theorem 2.1:

$$\|x^{k+1} - \xi\|_p \leq \phi(E(x_k))\,\|x^k - \xi\|_p \quad \text{and} \quad E(x^{k+1}) \leq \varphi(E(x^k)), \tag{20}$$

where the function ϕ is defined by (10) and $\varphi(t) = t\,\phi(t)$. Note that Han [17] proved the following estimate under the same initial condition:

$$E(x^{k+1}) \leq \frac{E(x^k)^2}{R(n,p)}$$

which according to Proinov ([18], Lemma 2.7) is a consequence of the second estimate in (20). Further, it is easy to see that $\phi(t) \sim (n-1)^{1/q}\,t$ as $t \to 0$. Then it follows from (13) and the first estimate in (20) that the asymptotic convergence rate of the Weierstrass iteration satisfies

$$\limsup_{k\to\infty} \frac{\|\,x^{k+1} - \xi\|_p}{\left\|\,x^k - \xi\right\|_p^2} \leq \frac{(n-1)^{1/q}}{\operatorname{sep}(f)}\,.$$

This estimate in the case $p = \infty$ is due to Hopkins et al. [8].

3. Preliminaries. Throughout this section J denotes an interval on $\mathbb{R}_+$ containing 0.

Definition 3.1 ([18])**.** A function $\varphi\colon J \to \mathbb{R}_+$ is said to be *quasi-homogeneous* of degree $r > 0$ on J if it satisfies the following condition:

$$\varphi(\lambda t) \leq \lambda^r \varphi(t) \quad \text{for all } \lambda \in (0,1) \text{ and } t \in J. \tag{21}$$

Definition 3.2 ([18])**.** A function $\varphi\colon J \to J$ is said to be a *gauge function of order* $r > 1$ on J if it is quasi-homogeneous of degree r on J and

$$\varphi(t) \leq t \quad \text{for all } t \in J. \tag{22}$$

A gauge function φ of order r on J is said to be a *strict gauge function* if the inequality (22) holds strictly whenever $t \in J\backslash\{0\}$.

Definition 3.3 ([18]). Let $T \colon D \subset X \to X$ be a map on an arbitrary set X. A function $E \colon D \to \mathbb{R}_+$ is said to be a *function of initial conditions* of T (with a gauge function φ on J) if there exists a function $\varphi \colon J \to J$ such that

$$E(Tx) \le \varphi(E(x)) \quad \text{for all } x \in D \text{ with } Tx \in D \text{ and } E(x) \in J. \tag{23}$$

Definition 3.4 ([18]). Let $T \colon D \subset X \to X$ be a map on an arbitrary set X, and let $E \colon D \to \mathbb{R}_+$ be a function of initial conditions of T with a gauge function on J. Then a point $x \in D$ is said to be an *initial point* of T if $E(x) \in J$ and all of the iterates $T^k x$ $(k = 0, 1, 2, \ldots)$ are well defined and belong to D.

The following theorem is a simplified version of Theorem 3.8 from [13].

Theorem 3.1 ([13]). *Let $T \colon D \subset X \to X$ be an operator in a normed space $(X, \|\cdot\|)$. Suppose that $E \colon D \to \mathbb{R}_+$ is a function of initial conditions of T with a strict gauge function φ of order $r > 1$ on J, and let ξ be a point in D such that $E(\xi) = 0$. Assume that T satisfies*

$$\|Tx - \xi\| \le \phi(E(x))\|x - \xi\| \quad \textit{for all } x \in D \textit{ with } E(x) \in J, \tag{24}$$

where $\phi \colon J \to \mathbb{R}_+$ is a nondecreasing function on J satisfying

$$\varphi(t) = t\,\phi(t) \quad \textit{for all } t \in J. \tag{25}$$

Then ξ is a unique fixed point of T in the set $U = \{x \in D : E(x) \in J\}$. Moreover, for each initial point x^0 of T the iterative sequence (6) *remains in the set U and converges to ξ with order r and error estimates*

$$\|x^{k+1} - \xi\| \le \lambda^{r^k}\,\|x^k - \xi\| \quad \textit{and} \quad \|x^k - \xi\| \le \lambda^{(r^k - 1)/(r-1)}\,\|x^0 - \xi\|, \tag{26}$$

where $\lambda = \phi(E(x^0))$.

Remark 3.1. A posteriori estimate in (26) was not formulated in work ([13], Theorem 3.8), but it follows immediately from (24) and the inequality $\phi(E(x^k)) \le \lambda^{r^k}$. The last inequality was proved in Proinov ([13], p. 44). Obviously, the a priori estimate in (26) is a consequence of the a posteriori estimate.

We end this section with a simple but useful sufficient condition for initial points.

Theorem 3.2 ([18]). *Let $T \colon D \subset X \to X$ be a map on a set X and $E \colon D \to \mathbb{R}_+$ a function of initial conditions of T with a gauge function φ on J. Suppose that*

$$x \in D \textit{ with } E(x) \in J \textit{ implies } Tx \in D.$$

Then every point $x_0 \in D$ such that $E(x_0) \in J$ is an initial point of T.

4. Proof of Theorem 2.1. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. In this section, we denote by T the *Weierstrass iteration function* $T\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ defined as follows:

$$Tx = x - W(x). \tag{27}$$

Obviously, the domain D of T is the set of all vectors in $\mathbb{K}^n$ with distinct components.

Lemma 4.1. *Let $u \in \mathbb{K}^n$ and $1 \leq p \leq \infty$. Then*

$$\left| \prod_{i=1}^{n} (1 + u_i) - 1 \right| \leq \left(1 + \frac{\|u\|_p}{n^{1/p}} \right)^n - 1. \tag{28}$$

Let $J = [0, R(n,p))$, where $R(n,p)$ is defined in (8).

Lemma 4.2. *Let $n \geq 2$ and $1 \leq p \leq \infty$. Then the real function*

$$\varphi(t) = \left[\left(1 + \frac{t}{(n-1)^{1/p}(1 - 2^{1/q}\, t)} \right)^{n-1} - 1 \right] t \tag{29}$$

is a strict gauge function of order $r = 2$ on J.

Lemma 4.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, ξ a root-vector of f and $1 \leq p \leq \infty$. Suppose that a vector $x \in \mathbb{K}^n$ satisfies*

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p < 1/2^{1/q}\,. \tag{30}$$

Then x has distinct components. Besides,

$$\|Tx - \xi\|_p \leq \phi(E(x))\, \|x - \xi\|_p \quad \text{and} \quad E(Tx) \leq \varphi(E(x)), \tag{31}$$

where the functions ϕ and φ are defined by (10) *and* (29)*, respectively.*

Proof. Using the triangle inequality, Hölder's inequality and the assumption (30), we get for $i \neq j$

$$\begin{aligned} |x_i - x_j| &\geq |\xi_i - \xi_j| - |x_i - \xi_i| - |x_j - \xi_j| \\ &\geq \left(1 - \left| \frac{x_i - \xi_i}{\xi_i - \xi_j} \right| - \left| \frac{x_j - \xi_j}{\xi_i - \xi_j} \right| \right) |\xi_i - \xi_j| \\ &\geq \left(1 - \left| \frac{x_i - \xi_i}{d_i(\xi)} \right| - \left| \frac{x_j - \xi_j}{d_j(\xi)} \right| \right) d_j(\xi) \\ &\geq \left(1 - 2^{1/q} \left\| \frac{x - \xi}{d(\xi)} \right\| \right) d_j(\xi) = \left(1 - 2^{1/q} E(x) \right) d_j(\xi) > 0 \end{aligned} \tag{32}$$

which means that x has distinct components. Now we shall prove the following inequalities:

$$|T_i(x) - \xi_i| \le \phi(E(x))|x_i - \xi_i| \qquad (i = 1, \dots, n). \tag{33}$$

From (2), taking into account that ξ is a root vector of f, we get

$$W_i(x) = (x_i - \xi_i) \prod_{j \ne i} \frac{x_i - \xi_j}{x_i - x_j} = (x_i - \xi_i) \prod_{j \ne i} (1 + u_j),$$

where $u_j = (x_j - \xi_j)/(x_i - x_j)$. Therefore,

$$T_i(x) - \xi_i = \left(\prod_{j \ne i} (1 + u_j) - 1 \right) (x_i - \xi_i).$$

Hence, using Lemma 4.1 we obtain

$$|T_i(x) - \xi_i| \le \left[\left(1 + \frac{\| u\|_p}{(n-1)^{1/p}} \right)^{n-1} - 1 \right] |x_i - \xi_i|, \tag{34}$$

where $u = (u_j)_{j \ne i} \in \mathbb{K}^{n-1}$. It follows from (32) that

$$|u_j| = \left| \frac{x_j - \xi_j}{x_i - x_j} \right| \le \frac{| x_j - \xi_j|}{(1 - 2^{1/q} E(x))\, d_j(\xi)} \le \frac{E(x)}{1 - 2^{1/q} E(x)} \tag{35}$$

which yields

$$\|u\|_p \le \frac{E(x)}{1 - 2^{1/q} E(x)}.$$

From (34) and the last inequality we obtain (33). Taking the p-norm in (33), we deduce the first inequality in (30). Dividing both sides of inequality (33) by $d_i(\xi)$ and taking the p-norm, we get the second inequality in (30). This completes the proof. □

Lemma 4.4. *The function E defined by* (7) *is a function of initial conditions of the Weierstrass iteration function T with a strict gauge function φ on J.*

Proof. The statement of the lemma follows from Lemmas 4.2 and 4.3. □

Lemma 4.5. *Every vector $x^0 \in \mathbb{K}^n$ such that $E(x^0) \in J$ is an initial point of the Weierstrass iteration function T.*

Proof. Let $x^0 \in \mathbb{K}^n$ be such that $E(x^0) \in J$. According to Lemma 4.3, $x^0 \in D$. Let $x \in D$ be such that $E(x) \in J$. Since $x \in D$, then $Tx \in \mathbb{K}^n$. By Lemma 4.4, we get $E(Tx) \le \varphi(E(x)) \le E(x)$. From this and $E(x) \in J$, we obtain $E(Tx) \in J$. Thus we have both $Tx \in \mathbb{K}^n$ and $E(Tx) \in J$. Applying Lemma 4.3 to Tx, we conclude that $Tx \in D$. According to Theorem 3.2, x^0 is an initial point of T. □

It follows from Lemmas 4.3, 4.4 and 4.5 that all the assumptions of Theorem 3.1 are satisfied for the Weierstrass iteration function T. Now applying Theorem 3.1 to the operator $T\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ we conclude that the Weierstrass iteration (1) is well defined and converges quadratically to ξ with error estimates (9). This completes the proof of Theorem 2.1.

REFERENCES

[1] Sendov Bl., A. Andreev, N. Kjurkchiev. Numerical Solution of Polynomial Equations, Handbook of Numerical Analysis (eds P. Ciarlet, J. Lions), Vol. **III**, Amsterdam, Elsevier, 1994, 625–778.

[2] Kyurkchiev N.V. Initial Approximations and Root Finding Methods, Mathematical Research, Vol. **104**, Berlin, Wiley, 1998.

[3] Petković M. Point Estimation of Root Finding Methods, Lecture Notes in Mathematics, Vol. **1933**, Berlin, Springer, 2008.

[4] Weierstrass K. Sitzungsberichte Königl. Akad. Wiss. Berlin, 1891, II, 1085–1101. Reproduced in: Ges. Werke, Vol. **3**, 1903, 251–269.

[5] Dochev K. Phys. Math. J. Bulg. Acad. Sci., **5**, 1962, 136–139 (in Bulgarian).

[6] Wang D., F. Zhao. J. Comput. Appl. Math., **38**, 1991, 447–456.

[7] Niell A. M. Comput. Math. Appl., **41**, 2001, 1–14.

[8] Hopkins M., B. Marshall, G. Schmidt, S. Zlobec. Z. Angew. Math. Mech., **74**, 1994, 295–306.

[9] Kjurkchiev N. V., S. M. Markov. Pliska Stud. Math. Bulg., **5**, 1983, 118–131.

[10] Sendov Bl., V. Popov. Méthodes d'Analyse Numérique, Tome **1**, Sofia, 1996.

[11] Yakoubsohn J.-C. Foundations of computational mathematics, River Edge, World Sci. Publ., 2002, 433–455.

[12] Smale S. The Merging of Disciplines: New Direction in Pure, Applied, and Computational Mathematics (eds R. E. Ewing, K. E. Gross, C. F. Martin), New York, Springer, 1986, 185–196.

[13] Proinov P. D. J. Complexity, **25**, 2009, 38–62.

[14] Proinov P. D., S. I. Ivanov. Scientific Researches of the Union of Scientists in Bulgaria–Plovdiv, Ser. B, **14**, 2012, 173–176.

[15] Cholakov S. I. Scientific Researches of the Union of Scientists in Bulgaria, Plovdiv, Ser. B, **14**, 2012, 197–200.

[16] Yakoubsohn J.-C. J. Complexity, **16**, 2000, 603–638.

[17] Han D. J. Comput. Math., **18**, 2000, 567–570.

[18] Proinov P. D. J. Complexity, **26**, 2010, 3–42.

Faculty of Mathematics and Informatics
University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: `proinov@uni-plovdiv.bg`
`petkoproinov@gmail.com`
`milenapetkova@uni-plovdiv.bg`
`milenapeeva@gmail.com`