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LOWER BOUNDS AND INFINITY CRITERION FOR BRAUER p-DIMENSIONS OF FINITELY-GENERATED FIELD EXTENSIONS

Ivan D. Chipchakov

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Abstract

Let E be a field, p a prime number and F/E a finitely-generated extension of transcendency degree t. This paper shows that if the absolute Galois group $\mathcal{G}_E$ is of nonzero cohomological p-dimension $\mathrm{cd}_p(E)$, then the field F has Brauer p-dimension $\mathrm{Brd}_p(F) \geq t$ except, possibly, in case $p = 2$, the Sylow pro-2-subgroups of $\mathcal{G}_E$ are of order 2, and F is a nonreal field. It announces that $\mathrm{Brd}_p(F)$ is infinite whenever $t \geq 1$ and the absolute Brauer p-dimension $\mathrm{abrd}_p(E)$ is infinite; moreover, for each pair (m, n) of integers with $1 \leq m \leq n$, there exists a central division F-algebra of exponent p^m and Schur index p^n.

Key words: Brauer group, Relative Brauer group, Schur index, Galois extension

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1. Introduction and index-exponent relations over finitely-generated field extensions. Let E be a field, $\mathrm{Br}(E)$ its Brauer group, $s(E)$ the class of finite-dimensional associative central simple E-algebras, and $d(E)$ the subclass of division algebras $D \in s(E)$. It is known that $\mathrm{Br}(E)$ is an abelian torsion group (cf. [16], Section 14.4), so it decomposes into the direct sum of its p-components $\mathrm{Br}(E)_p$, where p runs across the set $\mathbb{P}$ of prime numbers. Denote by $[A]$ the equivalence class in $\mathrm{Br}(E)$ of any $A \in s(E)$. The degree $\deg(A)$, the Schur index $\mathrm{ind}(A)$, and the exponent $\exp(A)$ (the order of $[A]$ in $\mathrm{Br}(E)$) are important invariants of A. Note that $\deg(A) = n.\mathrm{ind}(A)$, and $\mathrm{ind}(A)$ and $\exp(A)$ are related as follows (cf. [16], Sections 13.4, 14.4 and 15.2):

(1.1) $\exp(A)$ divides $\mathrm{ind}(A)$ and is divisible by every $p \in \mathbb{P}$ dividing $\mathrm{ind}(A)$. For each $B \in s(E)$ with $\mathrm{ind}(B)$ relatively prime to $\mathrm{ind}(A)$, $\mathrm{ind}(A \otimes_E B) = \mathrm{ind}(A).\mathrm{ind}(B)$; in particular, the tensor product $A \otimes_E B$ lies in $d(E)$, provided that $A \in d(E)$ and $B \in d(E)$.

As it is shown by Brauer, (1.1) fully describes the generally valid restrictions between Schur indices and exponents:

(1.2) Given a pair (m, n) of positive integers, such that $n \mid m$ and n is divisible by any $p \in \mathbb{P}$ dividing m, there is a field F and $D \in d(F)$ with $\mathrm{ind}(D) = m$ and $\exp(D) = n$ (Brauer, see [16], Section 19.6). One can take as F any rational (i.e. purely transcendental) extension of infinite transcendency degree over an arbitrary field F_0.

A field E is said to be of Brauer p-dimension $\mathrm{Brd}_p(E) = n$, where $n \in \mathbb{Z}$, if n is the least integer for which $\mathrm{ind}(D) \leq \exp(D)^n$ whenever $D \in d(E)$ and $[D] \in \mathrm{Br}(E)_p$. We say that $\mathrm{Brd}_p(E) = \infty$, if there exists a sequence $D_\nu \in d(E)$, $\nu \in \mathbb{N}$, such that $[D_\nu] \in \mathrm{Br}(E)_p$ and $\mathrm{ind}(D_\nu) > \exp(D_\nu)^\nu$, for each index ν. By an absolute Brauer p-dimension (abbr, $\mathrm{abrd}_p(E)$) of E, we mean the supremum $\sup\{\mathrm{Brd}_p(R)\colon\ R \in \mathrm{Fe}(E)\}$. Here and in the sequel, $\mathrm{Fe}(E)$ denotes the set of finite extensions of E in a separable closure E_{sep}. In what follows, we denote by $\mathrm{trd}(F/E)$ the transcendency degree and $I(F/E)$ stands for the set of intermediate fields of any extension F/E.

Clearly, $\mathrm{Brd}_p(E) \leq \mathrm{abrd}_p(E)$, for every field E and $p \in \mathbb{P}$. It is known that $\mathrm{Brd}_p(E) = \mathrm{abrd}_p(E) = 1$, for every $p \in \mathbb{P}$, in the following cases:

(1.3) (i) E is a global or local field (by class field theory, see, e.g., [2], Chs. VI and Ch. VII, by Serre and Tate, respectively);

(ii) E is the function field of an algebraic surface (defined) over an algebraically closed field E_0 [8, 12];

(iii) E is the function field of an algebraic curve over a pseudo algebraically closed field E_0 with $\mathrm{cd}_p(\mathcal{G}_{E_0}) > 0$ [5].

By a Brauer dimension and an absolute Brauer dimension of E, we mean the suprema $\mathrm{Brd}(E) = \sup\{\mathrm{Brd}_p(E)\colon p \in \mathbb{P}\}$ and $\mathrm{abrd}(E) = \sup\{\mathrm{abrd}_p(E) : p \in \mathbb{P}\}$, respectively. It would be of interest to know whether the function fields of algebraic varieties over a global, local or algebraically closed field are of finite absolute Brauer dimensions. Note also that fields of finite absolute Brauer p-dimensions, for all $p \in \mathbb{P}$, are singled out for their place in research areas like Galois cohomology (cf. [9], Section 3, [3], Remark 3.6, and [4], the end of Section 3 and Corollary 5.7) and the structure theory of their locally finite-dimensional central division algebras (see [3], Proposition 1.1 and the paragraph at the bottom of page 2). These observations draw one's attention to the following open problem:

(1.4) Find whether the class of fields E of finite absolute Brauer p-dimensions, for a fixed $p \in \mathbb{P}$ different from $\mathrm{char}(E)$, is closed under the formation of finitely-generated extensions.

The following result of [3] is used there for proving that the class of fields E with $\mathrm{Brd}(E) < \infty$ is not closed under taking finitely-generated extensions:

Theorem 1.1. *Let E be a field, $p \in \mathbb{P}$, and let F/E be a finitely-generated extension such that* $\mathrm{trd}(F/E) = t \geq 1$. *Then*:

(i) $\mathrm{Brd}_p(F) \geq \mathrm{abrd}_p(E) + t - 1$, *if* $\mathrm{abrd}_p(E) < \infty$ *and F/E is rational*;

(ii) *When* $\mathrm{abrd}_p(E) = \infty$, *there are* $\{D_{n,m} \in d(F)\colon\ n \in \mathbb{N}, m = 1, \dots, n\}$ *with* $\exp(D_{n,m}) = p^m$ *and* $\mathrm{ind}(D_{n,m}) = p^n$, *for each admissible pair (n, m)*;

(iii) *$Brd_p(F) = \infty$, provided $p = \mathrm{char}(E)$ and the degree $[E\colon E^p]$ is infinite, where $E^p = \{e^p\colon\ e \in E\}$; if $\mathrm{char}(E) = p$ and $[E\colon E^p] = p^{\nu} < \infty$, then $\nu + t - 1 \leq$ $\mathrm{Brd}_p(F) < \nu + t$.*

Theorem 1.1 is supplemented in [3], Section 3, as follows:

(1.5) Given a finitely-generated field extension F/E with $\mathrm{trd}(F/E) = t \geq 1$ and $\mathrm{abrd}_p(E) < \infty$ when p runs across some nonempty subset $P \subseteq \mathbb{P}$, there exists a finite subset $P(F/E)$ of P, such that $\mathrm{Brd}_p(F) \geq \mathrm{abrd}_p(E) + t - 1$, for each $p \in P \setminus P(F/E)$.

It is worth noting that there exist field extensions F/E satisfying the conditions of (1.5), for $P = \mathbb{P}$, such that $P(F/E)$ is necessarily nonempty.

Example. Let E be a real closed field, F the function field of the Brauer–Severi variety corresponding to the symbol E-algebra $A = A_{-1}(-1, -1; E)$, and $F' = F \otimes_E E(\sqrt{-1})$. By the Artin–Schreier theory (cf. [11], Ch. XI, Theorem 2), then $E(\sqrt{-1}) = E_{\mathrm{sep}}$, whence $\mathrm{abrd}_p(E) = 0$, for all $p \in \mathbb{P} \setminus \{2\}$. Since -1 does not lie in the norm group $N(E(\sqrt{-1})/E)$, it also follows that $A \in d(E)$. Note further that $\mathrm{trd}(F/E) = 1$, $[A \otimes_E F] = 0$ in $\mathrm{Br}(F)$, and $F'/E(\sqrt{-1})$ is a rational extension (see [18], Theorem 13.8 and Corollaries 13.9 and 13.16). In view of Tsen's theorem (cf. [16], Section 19.4), the noted property of F' ensures that it is a C_1-field, so it follows from [19], Ch. II, Proposition 6, that $\mathrm{cd}(\mathcal{G}_{F'}) \leq 1$. As $A \otimes_E F \cong A_1(-1,-1;F)$ over F, the equality $[A \otimes_E F] = 0$ implies that F is a nonreal field, so it follows from the Artin–Schreier theory that $\mathcal{G}_F$ is a torsion-free group. Finally, observing that $\mathcal{G}_{F'}$ embeds in $\mathcal{G}_F$ as an open subgroup, one obtains from [19], Ch. I, 4.2, Corollary 3, that $\mathrm{cd}(\mathcal{G}_F) \leq 1$, which means that $\mathrm{abrd}(F) = 0 < \mathrm{abrd}_2(E) = 1$.

Statement (1.1), Theorem 1.1 and basic properties of finitely-generated field extensions (cf. [11], Ch. X) imply the following:

(1.6) If the answer to (1.4) is affirmative for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(E)$, and each finitely-generated extension F/E with $\mathrm{trd}(F/E) = t \geq 1$, then there exists $c_t(p) \in \mathbb{N}$, such that $\mathrm{Brd}_p(\Phi) \leq c_t(p)$ whenever Φ/E is a finitely-generated extension and $\mathrm{trd}(\Phi/E) < t$ (see also [3], Proposition 4.6).

Theorem 1.1 (i) shows that the solution to [1], Problem 4.5, concerning the possibility to find a good definition of a field dimension $\dim(E)$, is negative except, possibly, in the case of $\mathrm{abrd}(E) < \infty$. In addition, it implies that if $\mathrm{abrd}(E) < \infty$ and [1], Problem 4.5, is solved affirmatively, for all finitely-generated extensions F/E, then the fields F satisfy much stronger conditions than the one conjectured by (1.6) (see [3], (1.5)). As to our next result (for a proof, see [3], Proposition 5.8), it indicates that the answer to (1.4) will be positive, for finitely-generated extensions F/E with $\mathrm{trd}(F/E) \leq n$, for some $n \in \mathbb{N}$, if this is the case in zero characteristic (see also [3], Remark 5.9, for an application of de Jong's theorem [8]):

(1.7) Let E be a field of characteristic $q > 0$ and F/E a finitely-generated extension. Then there exists a field E' with $\mathrm{char}(E) = 0$ and a finitely-generated extension F'/E' satisfying the following:

(i) $\mathcal{G}_{E'} \cong \mathcal{G}_E$ and $\mathrm{trd}(F'/E') = \mathrm{trd}(F/E)$;

(ii) $\mathrm{Brd}_p(F') \geq \mathrm{Brd}_p(F)$, $\mathrm{abrd}_p(F') \geq \mathrm{abrd}_p(F)$, $\mathrm{Brd}_p(E') = \mathrm{Brd}_p(E)$ and $\mathrm{abrd}_p(E') = \mathrm{abrd}_p(E)$, for each $p \in \mathbb{P}$ different from q.

The proof of Theorem 1.1 in [3] relies on the following two lemmas. When $\mu = 1$, the former one is a theorem due to Albert. Besides in [3], Section 3, a proof of the former lemma can be found in [15], Section 1.

Lemma 1.2. *A field E satisfies the inequality $\mathrm{abrd}_p(E) \leq \mu$, for some $p \in \mathbb{P}$ and $\mu \in \mathbb{N}$, if and only if, for each $E' \in \mathrm{Fe}(E)$, $\mathrm{ind}(\Delta_{E'}) \leq p^\mu$ whenever $\Delta_{E'} \in d(E')$ and $\exp(\Delta) = p$.*

Lemma 1.3. *Let K be a field, $F = K(X)$ a rational extension of K with $\mathrm{trd}(F/K) = 1$, $f(X) \in K[X]$ a separable and irreducible polynomial over K, L an extension of K in K_{sep} obtained by adjunction of a root of f, v a discrete valuation of F acting trivially on K with a uniform element f, and $(F_v, \bar{v})$ a Henselization of (F, v). Suppose that $\widetilde{D} \in d(L)$ is an algebra of exponent p. Then*

L is K-isomorphic to the residue field of $(F_v, \bar{v})$, and there exist $D' \in d(F_v)$ and $D \in d(F)$, such that $\exp(D) = \exp(D') = p$, *$[D \otimes_F F_v] = [D']$, and D' is an inertial lift of $\widetilde{D}$ over F_v.*

2. The main result. The purpose of this paper is to prove the following assertion which applied to a field with $\text{abrd}_p(E) = 0$, improves the inequality in Theorem 1.1 (i):

Theorem 2.1. *Let F be a finitely-generated extension of a field E with $\text{cd}_p(\mathcal{G}_E) \neq 0$. Then $\text{Brd}_p(F) \geq \text{trd}(F/E)$ except, possibly, when $p = 2$, the Sylow pro-2-subgroups of $\mathcal{G}_E$ are of order 2, and F is a nonreal field.*

The following result is contained in [3], Propositions 4.6 and 5.10, and is obtained by the method of proving Theorem 2.1 (see also [4], (4.10) and Proposition 4.3):

Theorem 2.2. *Assume that E is a field of type pointed out in* (1.3). *Then $\text{Brd}_p(F) \geq 1 + \text{trd}(F/E)$, for every finitely-generated extension F/E.*

Remark 2.3. (i) Theorem 2.1 ensures that $\text{Brd}_p(\Phi) \geq n$, $p \in \mathbb{P}$, if Φ is a finitely-generated extension of a quasifinite field Φ_0, and $\text{trd}(\Phi/\Phi_0) = n$. Therefore, one obtains following the proof of [3], Proposition 5.10, that the conclusion of Theorem 2.2 remains valid, if E is endowed with a Henselian discrete valuation whose residue field is quasifinite.

(ii) Given a finitely-generated field extension F/E with $\text{trd}(F/E) = k$, Theorem 2.1 implies Nakayama's inequalities $\text{Brd}_p(F) \geq k - 1$, $p \in \mathbb{P}$ (cf. [8], Section 2). When $\text{cd}_p(\mathcal{G}_E) = 0$, for some p, and E is perfect in the case of $p = \text{char}(E)$, we have $\text{Brd}_p(F) = k - 1$ if and only if this holds in the case where E is algebraically closed. The claim that $\text{Brd}(F) = k - 1$ when E is algebraically closed is the content of the so-called Standard Conjecture, for function fields of algebraic varieties over an algebraically closed field (see [12], Section 1, [13], page 3, and for relations with (1.4), the end of [3], Section 4).

The proof of Theorem 2.1 is based on the same idea as the one of Theorem 1.1. It relies on the following lemmas proved in [3].

Lemma 2.4. *Let (K, v) be a nontrivially real-valued field, and $(K_v, \bar{v})$ a Henselization of (K, v). Assume that $\Delta_v \in d(K_v)$ has exponent $p \in \mathbb{P}$. Then there exists $\Delta \in d(K)$, such that* $\exp(\Delta) = p$ *and* $[\Delta \otimes_K K_v] = [\Delta_v]$.

Lemma 2.4 is essentially due to Saltman [17], and our next lemma is a special case of the Grunwald–Wang theorem (cf. [14], Theorems 1 and 2).

Lemma 2.5. *Let F be a field, $S = \{v_1, \dots, v_s\}$ a finite set of non-equivalent nontrivial real-valued valuations of F, and for each index j, let F_{v_j} be a Henselization of K in K_{sep} relative to v_j, and L_j/F_{v_j} be a cyclic field extension of degree p^{μ_j}, for some $p \in P$ and $\mu_j \in \mathbb{N}$. Put* $\mu = \max\{\mu_1, \dots, \mu_s\}$ *and suppose that*

$\sqrt{-1} \in F$ *in case* $\mu \geq 3$, $p = 2$ *and* $\mathrm{char}(F) = 0$*. Then there exists a degree* p^μ *cyclic field extension* L/F*, such that* $L_{v'_j}$ *is* F_{v_j}*-isomorphic to* L_j*, where* v'_j *is a valuation of* L *extending* v_j*, for* $j = 1, \dots, s$*.*

In the rest of this Section, we recall some general results on Henselian valuations which are used (often implicitly, like Lemma 1.3) for proving Theorem 2.1. A Krull valuation v of a field K is called Henselian, if v extends uniquely, up-to an equivalence, to a valuation v_L on each algebraic extension L of K. Assuming that v is Henselian, denote by $v(L)$ the value group and by $\widehat{L}$ the residue field of (L, v_L). It is known that $\widehat{L}/\widehat{K}$ is an algebraic extension and $v(K)$ is a subgroup of $v(L)$. When L/K is finite and $e(L/K)$ is the index of $v(K)$ in $v(L)$, by Ostrowski's theorem [6], Theorem 17.2.1, $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$. In particular, if $\mathrm{char}(\widehat{K})$ does not divide $[L\colon K]$, then $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$. Ostrowski's theorem implies that there are group isomorphisms $v(K)/pv(K) \cong v(L)/pv(L)$, $p \in \mathbb{P}$, and in case $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$, they are canonically induced by the natural embedding of K into L.

As usual, a finite extension R of K is called inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}$ is separable over $\widehat{K}$. It follows from the Henselity of v that the compositum K_{ur} of inertial extensions of K in K_{sep} has the following properties:

(2.1) (i) $v(K_{\mathrm{ur}}) = v(K)$ and the finite extensions of K in K_{ur} are inertial;

(ii) Each finite extension of $\widehat{K}$ in $\widehat{K}_{\mathrm{sep}}$ is $\widehat{K}$-isomorphic to the residue field of an inertial extension of K; hence, $\widehat{K}_{\mathrm{ur}}$ is $\widehat{K}$-isomorphic to $\widehat{K}_{\mathrm{sep}}$;

(iii) K_{ur}/K is a Galois extension with $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$.

Similarly, it is known that each $\Delta \in d(K)$ has a unique, upto an equivalence, valuation v_Δ extending v so that the value group $v(\Delta)$ of (Δ, v_Δ) is abelian (see [7]). Note that $v(\Delta)$ includes $v(K)$ as an ordered subgroup of index $e(\Delta/K) \leq [\Delta\colon K]$, the residue division ring $\widehat{\Delta}$ of (Δ, v_Δ) is a $\widehat{K}$-algebra, and $[\widehat{\Delta}\colon \widehat{K}] \leq [\Delta\colon K]$. Moreover, by Ostrowski–Draxl's theorem (cf. [7], (1.2)), $e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}] \mid [\Delta\colon K]$, and in case $\mathrm{char}(\widehat{K}) \dagger [\Delta\colon K]$, $[\Delta\colon K] = e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D\colon K] = [\widehat{D}\colon \widehat{K}]$ and $\widehat{D} \in d(\widehat{K})$. In what follows, we also need the following results (see [7], Remark 3.4 and Theorems 2.8 and 3.1):

(2.2) (i) Each $\widetilde{D} \in d(\widehat{K})$ has a unique, up-to an F-isomorphism, inertial lift D over K(i.e. $D \in d(K)$, D is inertial over K and $\widehat{D} = \widetilde{D}$);

(ii) The set $\mathrm{IBr}(K)$ of Brauer equivalence classes of inertial K-algebras forms a subgroup of $\mathrm{Br}(K)$ canonically isomorphic to $\mathrm{Br}(\widehat{K})$;

(iii) For each $\Theta \in d(K)$ inertial over K, and any $R \in I(K_{\mathrm{ur}}/K)$, $[\Theta \otimes_K R] \in \mathrm{IBr}(R)$ and $\mathrm{ind}(\Theta \otimes_K R) = \mathrm{ind}(\widehat{\Theta} \otimes_{\widehat{K}} \widehat{R})$.

3. Proof of Theorem 2.1. Let E be a field with $\mathrm{cd}_p(\mathcal{G}_E) > 0$, for some $p \in \mathbb{P}$, and let F/E be a finitely-generated extension. Throughout this Section, E_{sep} is identified with its E-isomorphic copy in F_{sep}, and for any field Y, $r_p(Y)$ denotes the rank of the Galois group $\mathcal{G}(Y(p)/Y)$ of the maximal p-extension $Y(p)$ of Y (in Y_{sep}) as a pro-p-group. Assuming that $\mathrm{trd}(F/E) = t$ and G_p is a Sylow pro-p-subgroup of $\mathcal{G}_E$, we deduce Theorem 2.1 by proving the following:

(3.1) There exists $D \in d(F)$, such that $\exp(D) = p$, $\mathrm{ind}(D) = p^t$ and D is presentable as a tensor product of cyclic F-algebras of degree p except, possibly, in the case where $p = 2$, G_2 is of order 2 and F is a nonreal field.

Let E_p be the fixed field and $o(G_p)$ the order of G_p. Our assumptions show that $r_p(E_p) \geq 1$, which implies the existence of a field $M \in \mathrm{Fe}(E)$ with $r_p(M) \geq 1$ (apply the method of proving [16], Section 13.2, Proposition b). Moreover, M can be chosen to be nonreal unless $p = 2$ and $o(G_p) = 2$. Assuming that M is nonreal, one obtains from [20], Theorem 2, that there exists a $\mathbb{Z}_p$-extension Φ of M in E_{sep}. Hence, by Galois theory and the fact that $\mathbb{Z}_p$ is continuously isomorphic to its open subgroups, $\Phi M'/M'$ is Galois with $\mathcal{G}(\Phi M'/M') \cong \mathbb{Z}_p$, for each $M' \in \mathrm{Fe}(E)$. This makes it easy to obtain from basic properties of valuation prolongations on finite extensions that M can be chosen as an E-isomorphic copy of the residue field of a height t valuation v of F, trivial on E with $v(F) = \mathbb{Z}^t$. Here $\mathbb{Z}^t$ is viewed as an ordered abelian group with respect to the inversely-lexicographic ordering.

Let $(F_v, \bar{v})$ be a Henselization of (F, v). First suppose that $t = 1$ and take $\pi \in F$ so that $\langle v(\pi)\rangle = v(F)$. Then v lies in an infinite system of nonequivalent discrete valuations of F trivial on E (cf. [2], Ch. II, Lemma 3.1). In view of Lemma 2.5, this implies the existence of degree p cyclic extensions F_n, $n \in \mathbb{N}$, of F, such that F_1/F is inertial relative to v, and $F_n \subset F_v$, $n \geq 2$. Let φ_n be a generator of $\mathcal{G}(F_n/F)$, for each $n \in \mathbb{N}$. It follows from the choice of F_1 that the cyclic F-algebra $(F_1/F, \sigma_1, \pi)$ lies in $d(F)$ and $(F_1/F, \sigma_1, \pi) \otimes_F F_v \in d(F_v)$, which proves (3.1) in case $t = 1$.

Assume now that $t \geq 2$, and fix elements $\pi_1, \dots, \pi_t \in K$ so that $v(F)$ be generated by the set $\{v(\pi_j)\colon\ j = 1, \dots, t\}$, and $H = \langle v(\pi_1)\rangle$ be the minimal nontrivial isolated subgroup of $v(F)$. Then v and H induce canonically on F a valuation v_H with $v_H(F) = v(F)/H$; also, they give rise to a valuation $\hat{v}_H$ of the residue field F_H of (F, v_H) with $\hat{v}_H(F_H) = H$ and a residue field equal to M (cf. [6], Section 5.2). In addition, it is easily verified that F_H/E is a finitely-generated extension with $\mathrm{trd}(F_H/E) = 1$. Hence, by the proof of the already considered special case of Theorem 2.1, there exist $D_H \in d(F_H)$ and $\Psi_H \in I(F_{H,\mathrm{sep}}/F_H)$, such that $\mathrm{ind}(D_H) = p$, $D_H \otimes_{F_H} \Psi_H \in d(\Psi_H)$, and $I(\Psi_H/F_H)$ contains infinitely many degree p cyclic extensions of F_H. Now observing that v_H is of height $t-1$, and using repeatedly (2.2), Lemmas 2.4, 2.5 and [4], (3.1) (i), one proves that there exists a cyclic F-algebra $D' \in d(F)$, such that $\mathrm{ind}(D') = p$, $D' \otimes_F F_{v_H} \in d(F_{v_H})$ and $D' \otimes_F F_{v_H}$ is an inertial lift of D_H over a Henselization F_{v_H} of F relative to v_H.

Similarly, it can be deduced from (2.2) that each degree p cyclic extension of F_H is realizable as the residue field of an inertial cyclic degree p extension of F relative to v_H. This implies the existence of an inertial extension $(F', v'_H)/(F, v_H)$, such that $[F' \colon F] = [F'F_{v_H} : F_{v_H}] = p^{t-1}$, $D' \otimes_F F' \in d(F)$ and $F' = F_2 \dots F_t$, where F_i/F is a degree p cyclic extension of F, for $i = 2, \dots, t$. In view of Morandi's theorem (cf. [7], Proposition 1.4), it is now easy to construct an algebra $\Delta \in d(F)$, such that $\exp(\Delta) = p$, $\mathrm{ind}(\Delta) = p^{t-1}$, Δ is presentable as a tensor product of cyclic F-algebras of degree p, $\Delta \otimes_F F_{v_H} \in d(F_{v_H})$, $\Delta \otimes_F F_{v_H}$ is nicely semi-ramified over F_{v_H}, in the sense of [7], and $(D' \otimes_F \Delta) \otimes_F F_{v_H} \in d(F_{v_H})$. Therefore, $D' \otimes_F \Delta \in d(F)$, $\exp(D' \otimes_F \Delta) = p$ and $\mathrm{ind}(D' \otimes_F \Delta) = p^t$, which proves (3.1), under the hypothesis that $o(G_p) > 2$.

It remains to be seen that (3.1) holds when $p = 2$, F is formally real and $o(G_2) = 2$. By the Artin–Schreier theory, $o(G_2) = 2$ if and only if the fixed field E_2 is real closed. Our proof also relies on the following lemma.

Lemma 3.1. *Let E be a formally real field and F a finitely-generated extension of E with* $\mathrm{trd}(F/E) = 1$. *Then F is formally real if and only if it has a discrete valuation v trivial on E, whose residue field $\widehat{F}$ is formally real.*

Proof. It is known and easy to prove (cf. [10], Lemma 1) that if F is a nonreal field and ω is a discrete valuation of F trivial on E, then the residue field of (F, ω) is nonreal as well. Assume now that F is formally real, fix a real closure F' of F in F_{sep}, and put $E' = E_{\mathrm{sep}} \cap F'$. Observe that $E_{\mathrm{sep}}F'/F'$ is a Galois extension with $\mathcal{G}(E_{\mathrm{sep}}F'/F') \cong \mathcal{G}_{E'}$. Since, by the Artin–Schreier theory, $F_{\mathrm{sep}} = F'(\sqrt{-1}) = E_{\mathrm{sep}}F'$, this means that $E_{\mathrm{sep}} = E'(\sqrt{-1})$, whence, E' is a real closure of E in E_{sep}. Note also that $E'F/E'$ is finitely-generated, $\mathrm{trd}(E'F/E') = 1$ and $E'F \subseteq F'$, i.e. the extension $E'F/E'$ satisfies the conditions of Lemma 3.1. This enables one to deduce from [10], Theorem 6 and Proposition, that $E'F$ has a discrete valuation v' trivial on E' and with a residue field E'. It is now easy to see that the valuation v of F induced by v' has the properties required by Lemma 3.1. Specifically, $\widehat{F}$ is E-isomorphic to a finite extension of E in E'. □

We are now in a position to prove the remaining case of (3.1). Suppose first that $t = 1$, put $F_0 = E(X)$, for some $X \in F$ transcendental over E, and denote by Ω_0 the extension of F_0 in F_{sep} generated by the square roots of the totally positive elements of F_0 (i.e. those realizable over F_0 as finite sums of squares, see [11], Ch. XI, Proposition 2). Then $F\Omega_0$ is formally real, which implies $A_{-1}(-1, -1; \Omega) \in d(\Omega)$, for each $\Omega \in I(F\Omega_0/F_0)$, proving the assertion of (3.1). Note also that Ω_0/F_0 is an infinite Galois extension with $\mathcal{G}(\Omega_0/F_0)$ of exponent 2. This follows from Kummer theory and the fact that cosets $(X^2 + a^2)F_0^{*2}$, $a \in E^*$, generate an infinite subgroup of F_0^*/F_0^{*2}.

Assume now that $t \geq 2$, define F_0 and Ω_0 as above and denote by F_1 the algebraic closure of F_0 in F. Applying Lemma 3.1 and proceeding by induction on t, one concludes that F has valuation v trivial on F_1, such that $v(F) =$

$\mathbb{Z}^{t-1}$, v is of height $t-1$ and $\widehat{F}$ is a formally real finite extension of F_1. Fix a Henselization $(F_v, \bar{v})$ and an F_1-isomorphic copy F_1' of $\widehat{F}$ in $F_{\rm sep}$. It is easily verified that $F_1'\Omega_0$ is a formally real field and $F\Omega_0/F$ is a Galois extension. As v is of height $t-1$, one proves, using repeatedly (2.1) and Lemma 2.5, that $I(F\Omega_0/F)$ contains infinitely many quadratic and inertial extensions of F relative to v. Therefore, there exist fields $Y_n \in I(F\Omega_0/F)$, $n \in \mathbb{N}$, such that $[Y_n \colon F] = 2$, $[Y_1 \dots Y_n \colon F] = 2^n$ and $Y_1 \dots Y_n$ is inertial over F relative to v, for each index n. Fix a generator q_j of $\mathcal{G}(Y_j/F)$, and take elements $\pi_j \in F$, $j = 2, \dots, t$, so that $\langle v(\pi_2), \dots, v(\pi_t) \rangle = v(F)$. Put $\Delta_1 = A_{-1}(-1, -1; F)$ and consider the cyclic F-algebras $\Delta_j = (Y_j/F, q_j, \pi_j)$, $j = 2, \dots, t$. Since $F_1'\Omega_0$ is formally real, $A_{-1}(-1, -1; F) \otimes_F F_1'\Omega_0 \in d(F_1'\Omega_0)$, so it follows from Morandi's theorem, the noted properties of the fields Y_n, $n \in \mathbb{N}$, and the choice of $\pi_2, \dots, \pi_t$, that the F-algebra $\Delta = \Delta_1 \otimes_F \dots \otimes_F \Delta_t$ lies in $d(F)$ and $\Delta \otimes_F F_v \in d(F_v)$. This yields $\exp(\Delta) = 2$ and $\operatorname{ind}(\Delta) = 2^t$, so (3.1) and Theorem 2.1 are proved.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: chipchak@math.bas.bg