LOWER BOUNDS AND INFINITY CRITERION FOR BRAUER $p$-DIMENSIONS OF FINITELY-GENERATED FIELD EXTENSIONS

Ivan D. Chipchakov

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Abstract

Let $E$ be a field, $p$ a prime number and $F/E$ a finitely-generated extension of transcedency degree $t$. This paper shows that if the absolute Galois group $G_E$ is of nonzero cohomological $p$-dimension $cd_p(E)$, then the field $F$ has Brauer $p$-dimension $\text{Brd}_p(F) \geq t$ except, possibly, in case $p = 2$, the Sylow pro-$2$-subgroups of $G_E$ are of order $2$, and $F$ is a nonreal field. It announces that $\text{Brd}_p(F)$ is infinite whenever $t \geq 1$ and the absolute Brauer $p$-dimension $	ext{abrd}_p(E)$ is infinite; moreover, for each pair $(m, n)$ of integers with $1 \leq m \leq n$, there exists a central division $F$-algebra of exponent $p^m$ and Schur index $p^n$.

Key words: Brauer group, Relative Brauer group, Schur index, Galois extension

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1. Introduction and index-exponent relations over finitely-generated field extensions. Let $E$ be a field, $\text{Br}(E)$ its Brauer group, $s(E)$ the class of finite-dimensional associative central simple $E$-algebras, and $d(E)$ the subclass of division algebras $D \in s(E)$. It is known that $\text{Br}(E)$ is an abelian torsion group (cf. [16], Section 14.4), so it decomposes into the direct sum of its $p$-components $\text{Br}(E)_p$, where $p$ runs across the set $\mathbb{P}$ of prime numbers. Denote by $[A]$ the equivalence class in $\text{Br}(E)$ of any $A \in s(E)$. The degree $\deg(A)$, the Schur index $\text{ind}(A)$, and the exponent $\exp(A)$ (the order of $[A]$ in $\text{Br}(E)$) are important invariants of $A$. Note that $\deg(A) = n \cdot \text{ind}(A)$, and $\text{ind}(A)$ and $\exp(A)$ are related as follows (cf. [16], Sections 13.4, 14.4 and 15.2):

\begin{equation}
\tag{1.1} \exp(A) \mid \text{ind}(A) \text{ and is divisible by every } p \in \mathbb{P} \text{ dividing } \text{ind}(A).
\end{equation}

For each $B \in s(E)$ with $\text{ind}(B)$ relatively prime to $\text{ind}(A)$, $\text{ind}(A \otimes E B) = \text{ind}(A) \cdot \text{ind}(B)$; in particular, the tensor product $A \otimes E B$ lies in $d(E)$, provided that $A \in d(E)$ and $B \in d(E)$.

As it is shown by Brauer, (1.1) fully describes the generally valid restrictions between Schur indices and exponents:

\begin{equation}
\tag{1.2} \text{Given a pair } (m, n) \text{ of positive integers, such that } n \mid m \text{ and } n \text{ is divisible by any } p \in \mathbb{P} \text{ dividing } m, \text{ there is a field } F \text{ and } D \in d(F) \text{ with } \text{ind}(D) = m \text{ and } \exp(D) = n \text{ (Brauer, see [16], Section 19.6).}
\end{equation}

One can take as $F$ any rational (i.e. purely transcendental) extension of infinite transcendency degree over an arbitrary field $F_0$.

A field $E$ is said to be of Brauer $p$-dimension $\text{Brd}_p(E) = n$, where $n \in \mathbb{Z}$, if $n$ is the least integer for which $\text{ind}(D) \leq \exp(D)^n$ whenever $D \in d(E)$ and $[D] \in \text{Br}(E)_p$. We say that $\text{Brd}_p(E) = \infty$, if there exists a sequence $D_\nu \in d(E)$, $\nu \in \mathbb{N}$, such that $[D_\nu] \in \text{Br}(E)_p$ and $\text{ind}(D_\nu) \geq \exp(D_\nu)^\nu$, for each index $\nu$. By an absolute Brauer $p$-dimension (abbr, abrd$_p(E)$) of $E$, we mean the supremum $\sup\{\text{Brd}_p(R) : R \in \text{Fe}(E)\}$. Here and in the sequel, $\text{Fe}(E)$ denotes the set of finite extensions of $E$ in a separable closure $E_{\text{sep}}$. In what follows, we denote by $\text{trd}(F/E)$ the transcendency degree and $I(F/E)$ stands for the set of intermediate fields of any extension $F/E$.

Clearly, $\text{Brd}_p(E) \leq \text{abrd}_p(E)$, for every field $E$ and $p \in \mathbb{P}$. It is known that $\text{Brd}_p(E) = \text{abrd}_p(E) = 1$, for every $p \in \mathbb{P}$, in the following cases:

\begin{enumerate}
\item[(1.3)] \text{(i) } $E$ is a global or local field (by class field theory, see, e.g., [2], Chs. VI and Ch. VII, by Serre and Tate, respectively);
\item[(ii)] $E$ is the function field of an algebraic surface (defined) over an algebraically closed field $E_0$ [8, 12];
\item[(iii)] $E$ is the function field of an algebraic curve over a pseudo algebraically closed field $E_0$ with $\text{cd}_p(\mathcal{G}_{E_0}) > 0$ [5].
\end{enumerate}
By a Brauer dimension and an absolute Brauer dimension of $E$, we mean the suprema $\text{Brd}(E) = \sup\{\text{Brd}_p(E) : p \in \mathbb{P}\}$ and $\text{abrd}(E) = \sup\{\text{abrd}_p(E) : p \in \mathbb{P}\}$, respectively. It would be of interest to know whether the function fields of algebraic varieties over a global, local or algebraically closed field are of finite absolute Brauer dimensions. Note also that fields of finite absolute Brauer $p$-dimensions, for all $p \in \mathbb{P}$, are singled out for their place in research areas like Galois cohomology (cf. [9], Section 3, [3], Remark 3.6, and [4], the end of Section 3 and Corollary 5.7) and the structure theory of their locally finite-dimensional central division algebras (see [5], Proposition 1.1 and the paragraph at the bottom of page 2). These observations draw one’s attention to the following open problem:

(1.4) Find whether the class of fields $E$ of finite absolute Brauer $p$-dimensions, for a fixed $p \in \mathbb{P}$ different from $\text{char}(E)$, is closed under the formation of finitely-generated extensions.

The following result of [3], Section 3, as follows:

**Theorem 1.1.** Let $E$ be a field, $p \in \mathbb{P}$, and let $F/E$ be a finitely-generated extension such that $\text{trd}(F/E) = t \geq 1$. Then:

(i) $\text{Brd}_p(F) \geq \text{abrd}_p(E) + t - 1$, if $\text{abrd}_p(E) < \infty$ and $F/E$ is rational;

(ii) When $\text{abrd}_p(E) = \infty$, there are $\{D_{n,m} \in d(F) : n \in \mathbb{N}, m = 1, \ldots, n\}$ with $\exp(D_{n,m}) = p^m$ and $\text{ind}(D_{n,m}) = p^n$, for each admissible pair $(n,m)$;

(iii) $\text{Brd}_p(F) = \infty$, provided $p = \text{char}(E)$ and the degree $[E : E^p]$ is infinite, where $E^p = \{e^p : e \in E\}$; if $\text{char}(E) = p$ and $[E : E^p] = p^\nu < \infty$, then $\nu + t - 1 \leq \text{Brd}_p(F) < \nu + t$.

Theorem 1.1 is supplemented in [3], Section 3, as follows:

(1.5) Given a finitely-generated field extension $F/E$ with $\text{trd}(F/E) = t \geq 1$ and $\text{abrd}_p(E) < \infty$ when $p$ runs across some nonempty subset $P \subseteq \mathbb{P}$, there exists a finite subset $P(F/E)$ of $P$, such that $\text{Brd}_p(F) \geq \text{abrd}_p(E) + t - 1$, for each $p \in P \setminus P(F/E)$.

It is worth noting that there exist field extensions $F/E$ satisfying the conditions of (1.5), for $P = \mathbb{P}$, such that $P(F/E)$ is necessarily nonempty.

**Example.** Let $E$ be a real closed field, $F$ the function field of the Brauer–Severi variety corresponding to the symbol $E$-algebra $A = A_{-1}(-1, -1; E)$, and $F' = F \otimes_E E(\sqrt{-1})$. By the Artin–Schreier theory (cf. [11], Ch. XI, Theorem 2), then $E(\sqrt{-1}) = E_{\text{sep}}$, whence $\text{abrd}_p(E) = 0$, for all $p \in \mathbb{P} \setminus \{2\}$. Since $-1$ does not lie in the norm group $N(E(\sqrt{-1})/E)$, it also follows that $A \in d(E)$. Note further that $\text{trd}(F/E) = 1$, $[A \otimes_E F] = 0$ in $\text{Br}(F)$, and $F'/E(\sqrt{-1})$ is a rational extension (see [18], Theorem 13.8 and Corollaries 13.9 and 13.16). In view of Tsen’s theorem (cf. [16], Section 19.4), the noted property of $F'$ ensures that it.
is a $C_1$-field, so it follows from [19], Ch. II, Proposition 6, that $\text{cd}(G_F) \leq 1$. As $A \otimes_E F \cong A_1(-1, -1; F)$ over $F$, the equality $[A \otimes_E F] = 0$ implies that $F$ is a nonreal field, so it follows from the Artin–Schreier theory that $G_F$ is a torsion-free group. Finally, observing that $G_{F'}$ embeds in $G_F$ as an open subgroup, one obtains from [19], Ch. I, 4.2, Corollary 3, that $\text{cd}(G_F) \leq 1$, which means that $\text{abr}(F) = 0 < \text{abr}(E) = 1$.

Statement (1.1), Theorem 1.1 and basic properties of finitely-generated field extensions (cf. [11], Ch. X) imply the following:

(1.6) If the answer to (1.4) is affirmative for some $p \in \mathbb{P}$, $p \not= \text{char}(E)$, and each finitely-generated extension $F/E$ with $\text{trd}(F/E) = t \geq 1$, then there exists $c_1(p) \in \mathbb{N}$, such that $\text{Br}(F) \leq c_1(p)$ whenever $F/E$ is a finitely-generated extension and $\text{trd}(F/E) < t$ (see also [8], Proposition 4.6).

Theorem 1.1 (i) shows that the solution to [1], Problem 4.5, concerning the possibility to find a good definition of a field dimension $\text{dim}(E)$, is negative except, possibly, in the case of $\text{abr}(E) < \infty$. In addition, it implies that if $\text{abr}(E) < \infty$ and [1], Problem 4.5, is solved affirmatively, for all finitely-generated extensions $F/E$, then the fields $F$ satisfy much stronger conditions than the one conjectured by (1.6) (see [3], (1.5)). As to our next result (for a proof, see [3], Proposition 5.8), it indicates that the answer to (1.4) will be positive, for finitely-generated extensions $F/E$ with $\text{trd}(F/E) \leq n$, for some $n \in \mathbb{N}$, if this is the case in zero characteristic (see also [3], Remark 5.9, for an application of de Jong’s theorem [8]):

(1.7) Let $E$ be a field of characteristic $q > 0$ and $F/E$ a finitely-generated extension. Then there exists a field $E'$ with $\text{char}(E) = 0$ and a finitely-generated extension $F'/E'$ satisfying the following:

(i) $G_{E'} \cong G_E$ and $\text{trd}(F'/E') = \text{trd}(F/E)$;

(ii) $\text{Br}(F) \geq \text{Br}(F')$, $\text{abr}(F') \geq \text{abr}(F)$, $\text{Br}(E') = \text{Br}(E)$ and $\text{abr}(E') = \text{abr}(E)$, for each $p \in \mathbb{P}$ different from $q$.

The proof of Theorem 1.1 in [3] relies on the following two lemmas. When $\mu = 1$, the former one is a theorem due to Albert. Besides in [3], Section 3, a proof of the former lemma can be found in [15], Section 1.

**Lemma 1.2.** A field $E$ satisfies the inequality $\text{abr}(E) \leq \mu$, for some $p \in \mathbb{P}$ and $\mu \in \mathbb{N}$, if and only if, for each $E' \in \text{Fe}(E)$, $\text{ind}(\Delta_{E'}) \leq p^\mu$ whenever $\Delta_{E'} \in d(E')$ and $\exp(\Delta) = p$.

**Lemma 1.3.** Let $K$ be a field, $F = K(X)$ a rational extension of $K$ with $\text{trd}(F/K) = 1$, $f(X) \in K[X]$ a separable and irreducible polynomial over $K$, $L$ an extension of $K$ in $K_{sep}$ obtained by adjunction of a root of $f$, $v$ a discrete valuation of $F$ acting trivially on $K$ with a uniform element $f$, and $(F_v, \bar{v})$ a Henselization of $(F, v)$. Suppose that $\bar{D} \in d(L)$ is an algebra of exponent $p$. Then
L is $K$-isomorphic to the residue field of $(F_v, \bar{v})$, and there exist $D' \in d(F_v)$ and $D \in d(F)$, such that $\exp(D) = \exp(D') = p$, $[D \otimes_F F_v] = [D']$, and $D'$ is an inertial lift of $D$ over $F_v$.

2. The main result. The purpose of this paper is to prove the following assertion which applied to a field with $\operatorname{abrd}_p(E) = 0$, improves the inequality in Theorem 1.1 (i):

**Theorem 2.1.** Let $F$ be a finitely-generated extension of a field $E$ with $\operatorname{cd}_p(G_E) \neq 0$. Then $\operatorname{Brd}_p(F) \geq \operatorname{trd}(F/E)$ except, possibly, when $p = 2$, the Sylow pro-$2$-subgroups of $G_E$ are of order $2$, and $F$ is a nonreal field.

The following result is contained in [3], Propositions 4.6 and 5.10, and is obtained by the method of proving Theorem 2.1 (see also [4], (4.10) and Proposition 4.3):

**Theorem 2.2.** Assume that $E$ is a field of type pointed out in (1.3). Then $\operatorname{Brd}_p(F) \geq 1 + \operatorname{trd}(F/E)$, for every finitely-generated extension $F/E$.

**Remark 2.3.** (i) Theorem 2.1 ensures that $\operatorname{Brd}_p(\Phi) \geq n$, $p \in \mathbb{P}$, if $\Phi$ is a finitely-generated extension of a quasifinite field $\Phi_0$, and $\operatorname{trd}(\Phi/\Phi_0) = n$. Therefore, one obtains following the proof of [3], Proposition 5.10, that the conclusion of Theorem 2.2 remains valid, if $E$ is endowed with a Henselian discrete valuation whose residue field is quasifinite.

(ii) Given a finitely-generated field extension $F/E$ with $\operatorname{trd}(F/E) = k$, Theorem 2.1 implies Nakayama’s inequalities $\operatorname{Brd}_p(F) \geq k - 1$, $p \in \mathbb{P}$ (cf. [8], Section 2). When $\operatorname{cd}_p(G_E) = 0$, for some $p$, and $E$ is perfect in the case of $p = \operatorname{char}(E)$, we have $\operatorname{Brd}_p(F) = k - 1$ if and only if this holds in the case where $E$ is algebraically closed. The claim that $\operatorname{Brd}(F) = k - 1$ when $E$ is algebraically closed is the content of the so-called Standard Conjecture, for function fields of algebraic varieties over an algebraically closed field (see [12], Section 1, [13], page 3, and for relations with (1.4), the end of [3], Section 4).

The proof of Theorem 2.1 is based on the same idea as the one of Theorem 1.1. It relies on the following lemmas proved in [3].

**Lemma 2.4.** Let $(K, v)$ be a nontrivially real-valued field, and $(K_v, \bar{v})$ a Henselization of $(K, v)$. Assume that $\Delta \in d(K_v)$ has exponent $p \in \mathbb{P}$. Then there exists $\Delta \in d(K)$, such that $\exp(\Delta) = p$ and $[\Delta \otimes_K K_v] = [\Delta_v]$.

Lemma 2.4 is essentially due to Saltman [17], and our next lemma is a special case of the Grunwald–Wang theorem (cf. [14], Theorems 1 and 2).

**Lemma 2.5.** Let $F$ be a field, $S = \{v_1, \ldots, v_s\}$ a finite set of non-equivalent nontrivial real-valued valuations of $F$, and for each index $j$, let $F_{v_j}$ be a Henselization of $K$ in $K_{sep}$ relative to $v_j$, and $L_j/F_{v_j}$ be a cyclic field extension of degree $p^{\mu_j}$, for some $p \in \mathbb{P}$ and $\mu_j \in \mathbb{N}$. Put $\mu = \max\{\mu_1, \ldots, \mu_s\}$ and suppose that

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\[\sqrt{-1} \in F \text{ in case } \mu \geq 3, \ p = 2 \text{ and } \text{char}(F) = 0. \] Then there exists a degree \(p^n\) cyclic field extension \(L/F\), such that \(L_{v_j}\) is \(F_{v_j}\)-isomorphic to \(L_j\), where \(v_j\) is a valuation of \(L\) extending \(v_j\), for \(j = 1, \ldots, s\).

In the rest of this Section, we recall some general results on Henselian valuations which are used (often implicitly, like Lemma 1.3) for proving Theorem 2.1. A Krull valuation \(v\) of a field \(K\) is called Henselian, if \(v\) extends uniquely, up-to an equivalence, to a valuation \(v_L\) on each algebraic extension \(L\) of \(K\). Assuming that \(v\) is Henselian, denote by \(v(L)\) the value group and by \(\hat{L}\) the residue field of \((L, v_L)\). It is known that \(\hat{L}/\hat{K}\) is an algebraic extension and \(v(\hat{K})\) is a subgroup of \(v(\hat{L})\). When \(L/K\) is finite and \(e(L/K)\) is the index of \((v(\hat{K})\) in \(v(\hat{L})\), by Ostrowski’s theorem \([9]\), Theorem 17.2.1, \([\hat{L}: \hat{K}]e(\hat{L}/\hat{K})\) divides \([L: K]\) and \([L: K][\hat{L}: \hat{K}]^{-1}e(\hat{L}/\hat{K})^{-1}\) is not divisible by any \(p \in \mathbb{P}, \ p \neq \text{char}(\hat{K})\). In particular, if char(\(\hat{K}\)) does not divide \([L: K]\), then \([L: K] = [\hat{L}: \hat{K}]e(\hat{L}/\hat{K})\). Ostrowski’s theorem implies that there are group isomorphisms \(v(\hat{K})/pv(\hat{K}) \cong v(\hat{L})/pv(\hat{L}), \ p \in \mathbb{P}\), and in case char(\(\hat{K}\)) \(\neq [L: K]\), they are canonically induced by the natural embedding of \(K\) into \(L\).

As usual, a finite extension \(R\) of \(K\) is called inertial, if \([R: K] = [\hat{R}: \hat{K}]\) and \(\hat{R}\) is separable over \(\hat{K}\). It follows from the Henselity of \(v\) that the compositum \(K_{ur}\) of inertial extensions of \(K\) in \(K_{sep}\) has the following properties:

1. \(v(K_{ur}) = v(K)\) and the finite extensions of \(K\) in \(K_{ur}\) are inertial;
2. Each finite extension of \(\hat{K}\) in \(\hat{K}_{sep}\) is \(\hat{K}\)-isomorphic to the residue field of an inertial extension of \(K\); hence, \(\hat{K}_{ur}\) is \(\hat{K}\)-isomorphic to \(\hat{K}_{sep}\);
3. \(K_{ur}/K\) is a Galois extension with \(G(K_{ur}/K) \cong G_{\hat{K}}\).

Similarly, it is known that each \(\Delta \in d(K)\) has a unique, up-to an equivalence, valuation \(v_\Delta\) extending \(v\) so that the value group \(v(\Delta)\) of \((\Delta, v_\Delta)\) is abelian (see \([7]\)). Note that \(v(\Delta)\) includes \(v(\Delta)\) as an ordered subgroup of index \(e(\Delta/K)\) \(\leq [\Delta: K]\), the residue division ring \(\hat{\Delta}\) of \((\Delta, v_\Delta)\) is a \(\hat{K}\)-algebra, and \([\Delta: K] \leq [\Delta: \hat{K}]\). Moreover, by Ostrowski–Draxl’s theorem (cf. \([7]\), (1.2)), \(e(\Delta/K)[\hat{\Delta}: \hat{K}] = [\Delta: K]\). In what follows, we also need the following results (see \([7]\), Remark 3.4 and Theorems 2.8 and 3.1):

1. Each \(\tilde{D} \in d(\hat{K})\) has a unique, up-to an \(F\)-isomorphism, inertial lift \(D\) over \(K\) (i.e. \(D \in d(K)\), \(D\) is inertial over \(K\) and \(\widetilde{D} = \tilde{D}\));
2. The set \(\text{IBr}(K)\) of Brauer equivalence classes of inertial \(K\)-algebras forms a subgroup of \(\text{Br}(K)\) canonically isomorphic to \(\text{Br}(\hat{K})\);
3. For each \(\Theta \in d(K)\) inertial over \(K\), and any \(R \in I(K_{ur}/K)\), \([\Theta \otimes_K R] \in \text{IBr}(R)\) and \(\text{ind}(\Theta \otimes_K R) = \text{ind}(\Theta \otimes_{\hat{K}} \hat{R})\).

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3. Proof of Theorem 2.1. Let $E$ be a field with $\text{cd}_p(G_E) > 0$, for some $p \in \mathbb{P}$, and let $F/E$ be a finitely-generated extension. Throughout this Section, $E_{\text{sep}}$ is identified with its $E$-isomorphic copy in $F_{\text{sep}}$, and for any field $Y$, $r_p(Y)$ denotes the rank of the Galois group $\mathcal{G}(Y(p)/Y)$ of the maximal $p$-extension $Y(p)$ of $Y$ (in $Y_{\text{sep}}$) as a pro-$p$-group. Assuming that $\text{trd}(F/E) = t$ and $G_p$ is a Sylow pro-$p$-subgroup of $\mathcal{G}_E$, we deduce Theorem 2.1 by proving the following:

(3.1) There exists $D \in d(F)$, such that $\exp(D) = p$, $\text{ind}(D) = p^t$ and $D$ is presentable as a tensor product of cyclic $F$-algebras of degree $p$ except, possibly, in the case where $p = 2$, $G_2$ is of order 2 and $F$ is a nonreal field.

Let $E_p$ be the fixed field and $o(G_p)$ the order of $G_p$. Our assumptions show that $r_p(E_p) \geq 1$, which implies the existence of a field $M \in \text{Fe}(E)$ with $r_p(M) \geq 1$ (apply the method of proving [16], Section 13.2, Proposition b). Moreover, $M$ can be chosen to be nonreal unless $p = 2$ and $o(G_p) = 2$. Assuming that $M$ is nonreal, one obtains from [20], Theorem 2, that there exists a $\mathbb{Z}_p$-extension $\Phi$ of $M$ in $E_{\text{sep}}$. Hence, by Galois theory and the fact that $\mathbb{Z}_p$ is continuously isomorphic to its open subgroups, $\Phi M'/M'$ is Galois with $\mathcal{G}(\Phi M'/M') \cong \mathbb{Z}_p$, for each $M' \in \text{Fe}(E)$. This makes it easy to obtain from basic properties of valuation prolongations on finite extensions that $M$ can be chosen as an $E$-isomorphic copy of the residue field of a height $t$ valuation $v$ of $F$, trivial on $E$ with $v(F) = \mathbb{Z}^t$. Here $\mathbb{Z}^t$ is viewed as an ordered abelian group with respect to the inversely-lexicographic ordering.

Let $(F_v, \bar{v})$ be a Henselization of $(F, v)$. First suppose that $t = 1$ and take $\pi \in F$ so that $(v(\pi)) = v(F)$. Then $v$ lies in an infinite system of nonequivalent discrete valuations of $F$ trivial on $E$ (cf. [2], Ch. II, Lemma 3.1). In view of Lemma 2.5, this implies the existence of degree $p$ cyclic extensions $F_n$, $n \in \mathbb{N}$, of $F$, such that $F_1/F$ is inertial relative to $v$, and $F_n \subset F_v$, $n \geq 2$. Let $\varphi_n$ be a generator of $\mathcal{G}(F_n/F)$, for each $n \in \mathbb{N}$. It follows from the choice of $F_1$ that the cyclic $F$-algebra $(F_1/F, \sigma_1, \pi)$ lies in $d(F)$ and $(F_1/F, \sigma_1, \pi) \otimes_F F_v \in d(F_v)$, which proves (3.1) in case $t = 1$.

Assume now that $t \geq 2$, and fix elements $\pi_1, \ldots, \pi_t \in K$ so that $v(F)$ be generated by the set $\{v(\pi_j) : j = 1, \ldots, t\}$, and $H = \langle v(\pi_1) \rangle$ be the minimal nontrivial isolated subgroup of $v(F)$. Then $v$ and $H$ induce canonically on $F$ a valuation $v_H$ with $v_H(F) = v(F)/H$; also, they give rise to a valuation $\hat{v}_H$ of the residue field $F_H$ of $(F, v_H)$ with $\hat{v}_H(F_H) = H$ and a residue field equal to $M$ (cf. [6], Section 5.2). In addition, it is easily verified that $F_H/E$ is a finitely-generated extension with $\text{trd}(F_H/E) = 1$. Hence, by the proof of the already considered special case of Theorem 2.1, there exist $D_H \in d(F_H)$ and $\Psi_H \in I(F_{H, \text{sep}}/F_H)$, such that $\text{ind}(D_H) = p$, $D_H \otimes_{F_H} \Psi_H \in d(\Psi_H)$, and $I(\Psi_H/F_H)$ contains infinitely many degree $p$ cyclic extensions of $F_H$. Now observing that $v_H$ is of height $t - 1$, and using repeatedly (2.2), Lemmas 2.4, 2.5 and [6], (3.1) (i), one proves that there exists a cyclic $F$-algebra $D' \in d(F)$, such that $\text{ind}(D') = p$, $D' \otimes_F F_{v_H} \in d(F_{v_H})$ and $D' \otimes_F F_{v_H}$ is an inertial lift of $D_H$ over a Henselization $F_{v_H}$ of $F$ relative to $v_H$.

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Similarly, it can be deduced from (2.2) that each degree $p$ cyclic extension of $F_H$ is realizable as the residue field of an inertial cyclic degree $p$ extension of $F$ relative to $v_H$. This implies the existence of an inertial extension $(F', v'_H)/(F, v_H)$, such that $[F': F] = [F'_H:v_H] = p^{t-1}$, $D' \otimes_F F' \in d(F)$ and $F' = F_2 \ldots F_t$, where $F_i/F$ is a degree $p$ cyclic extension of $F$, for $i = 2, \ldots, t$. In view of Morandi’s theorem (cf. [7], Proposition 1.4), it is now easy to construct an algebra $\Delta \in d(F)$, such that $\exp(\Delta) = p$, $\ind(\Delta) = p^{t-1}$, $\Delta$ is presentable as a tensor product of cyclic $F$-algebras of degree $p$, $\Delta \otimes_F F_{v_H} \in d(F_{v_H})$, $\Delta \otimes_F F'_{v_H}$ is nicely semi-ramified over $F_{v_H}$, in the sense of [7], and $(D' \otimes_F \Delta) \otimes_F F_{v_H} \in d(F_{v_H})$. Therefore, $D' \otimes_F \Delta \in d(F)$, $\exp(D' \otimes_F \Delta) = p$ and $\ind(D' \otimes_F \Delta) = p^t$, which proves (3.1), under the hypothesis that $o(G_p) > 2$.

It remains to see that (3.1) holds when $p = 2$, $F$ is formally real and $o(G_2) = 2$. By the Artin–Schreier theory, $o(G_2) = 2$ if and only if the fixed field $E_2$ is real closed. Our proof also relies on the following lemma.

**Lemma 3.1.** Let $E$ be a formally real field and $F$ a finitely-generated extension of $E$ with $\text{trd}(F/E) = 1$. Then $F$ is formally real if and only if it has a discrete valuation $v$ trivial on $E$, whose residue field $\tilde{F}$ is formally real.

**Proof.** It is known and easy to prove (cf. [10], Lemma 1) that if $F$ is a nonreal field and $\omega$ is a discrete valuation of $F$ trivial on $E$, then the residue field of $(F, \omega)$ is nonreal as well. Assume now that $F$ is formally real, fix a real closure $F'$ of $F$ in $F_{\text{sep}}$, and put $E' = E_{\text{sep}} \cap F'$. Observe that $E_{\text{sep}} F'/F'$ is a Galois extension with $G(E_{\text{sep}} F'/F') \cong G_E$. Since, by the Artin–Schreier theory, $F_{\text{sep}} = F'((\sqrt{-1}) = E_{\text{sep}} F'$, this means that $E_{\text{sep}} = E'((\sqrt{-1})$, whence, $E'$ is a real closure of $E$ in $F_{\text{sep}}$. Note also that $E' F/E'$ is finitely-generated, $\text{trd}(E' F/E') = 1$ and $E'F \subseteq F'$, i.e. the extension $E'F/E'$ satisfies the conditions of Lemma 3.1. This enables one to deduce from [10], Theorem 6 and Proposition, that $E'F$ has a discrete valuation $v'$ trivial on $E'$ and with a residue field $E'$. It is now easy to see that the valuation $v$ of $F$ induced by $v'$ has the properties required by Lemma 3.1. Specifically, $\tilde{F}$ is $E$-isomorphic to a finite extension of $E$ in $E'$.

We are now in a position to prove the remaining case of (3.1). Suppose first that $t = 1$, put $F_0 = E(X)$, for some $X \in F$ transcendental over $E$, and denote by $\Omega_0$ the extension of $F_0$ in $F_{\text{sep}}$ generated by the square roots of the totally positive elements of $F_0$ (i.e. those realizable over $F_0$ as finite sums of squares, see [11], Ch. XI, Proposition 2). Then $F \Omega_0$ is formally real, which implies $A_{-1}(-1, -1; \Omega) \in d(\Omega)$, for each $\Omega \in I(F \Omega_0/F_0)$, proving the assertion of (3.1). Note also that $\Omega_0/F_0$ is an infinite Galois extension with $G(\Omega_0/F_0)$ of exponent 2. This follows from Kummer theory and the fact that cosets $(X^2 + a^2)F_0^{a^2}$, $a \in E^*$, generate an infinite subgroup of $F_0^{a^2} / F_0^{a^2}$.

Assume now that $t \geq 2$, define $F_0$ and $\Omega_0$ as above and denote by $F_1$ the algebraic closure of $F_0$ in $F$. Applying Lemma 3.1 and proceeding by induction on $t$, one concludes that $F$ has valuation $v$ trivial on $F_1$, such that $v(F) = 930$

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$Z^{t-1}, \; v$ is of height $t - 1$ and $\hat{F}$ is a formally real finite extension of $F_1$. Fix a Henselization $(F_v, \hat{v})$ and an $F_1$-isomorphic copy $F'_1$ of $\hat{F}$ in $F_{sep}$. It is easily verified that $F'_1\Omega_0$ is a formally real field and $F\Omega_0/F$ is a Galois extension. As $v$ is of height $t - 1$, one proves, using repeatedly (2.1) and Lemma 2.5, that $I(F\Omega_0/F)$ contains infinitely many quadratic and inertial extensions of $F$ relative to $v$. Therefore, there exist fields $Y_n \in I(F\Omega_0/F)$, $n \in \mathbb{N}$, such that $[Y_n : F] = 2$, $[Y_1 \ldots Y_n : F] = 2^n$ and $Y_1 \ldots Y_n$ is inertial over $F$ relative to $v$, for each index $n$. Fix a generator $q_j$ of $G(Y_j/F)$, and take elements $\pi_j \in F$, $j = 2, \ldots, t$, so that $\langle v(\pi_2), \ldots, v(\pi_t) \rangle = v(F)$. Put $\Delta_1 = A_{-1}(-1, -1; F)$ and consider the cyclic $F$-algebras $\Delta_j = (Y_j/F, q_j, \pi_j)$, $j = 2, \ldots, t$. Since $F'_1\Omega_0$ is formally real, $A_{-1}(-1, -1; F) \otimes_F F'_1\Omega_0 \in d(F'_1\Omega_0)$, so it follows from Morandi’s theorem, the noted properties of the fields $Y_n$, $n \in \mathbb{N}$, and the choice of $\pi_2, \ldots, \pi_t$, that the $F$-algebra $\Delta = \Delta_1 \otimes_F \cdots \otimes_F \Delta_t$ lies in $d(F)$ and $\Delta \otimes_F F_v \in d(F_v)$. This yields $\exp(\Delta) = 2$ and $\text{ind}(\Delta) = 2^t$, so (3.1) and Theorem 2.1 are proved.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: chipchak@math.bas.bg