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ALL COREGULAR SEMIGROUPS IN THE IDEAL $K(n, 2)$ OF THE FULL TRANSFORMATION SEMIGROUP T_n

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Abstract

The semigroup S is called coregular if $a^3 = a$ for all $a \in S$. In this paper, we give a characterization of all coregular transformation semigroups within the ideal $K(n, 2)$ of all transformations on an n-element set with rank less or equal to two.
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1. Introduction. The description of varieties, where periodic semigroups are locally finite (i.e. semigroups which are finite whenever they are finite generated), plays an important role in the study of algorithmic problems in semigroup varieties. An overview about Burnside-type problems in semigroup varieties was published by O. G. Kharlampovich and M. V. Sapir [8].

In 1952, J. A. Green and D. Rees [6] considered equations of the form $x \approx x^k$. The authors proved that a semigroup satisfying $x \approx x^k$ is locally finite if and only if all its subgroups are locally finite. In particular, a variety given by an identity $x \approx x^k$ is locally finite if $n = 2, 3, 4, 5, 7$ because every group satisfying this identity is locally finite. A semigroup satisfying $x \approx x^k$ is a union of groups. Semigroups which are unions of groups are called completely regular.

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It is well known that each semigroup is an isomorphic image of a subsemigroup of the full transformation semigroup on a suitable set. If a semigroup S has a homomorphism onto a locally finite semigroup and every pre-image of an idempotent is a locally finite semigroup, then S is locally finite [4]. This motivates a study of transformation semigroups, satisfying an equation $x \approx x^k$.

Denote by T_n the full transformation semigroup on the set $\{1, \ldots, n\}$. It is well known that the semigroup $K(n, d)$ in which each transformation has an image with size less or equal to d is an ideal of T_n. Let S be a subsemigroup of $K(n, d)$ in which each element α satisfies the equation $x \approx x^k$. Without loss of generality we can assume that k is minimal with this property. Then, α restricted to its image imα is a bijection and can be written as a product of independent cycles of length at most d. This shows that the order of α devises the least common multiple of the length's of the cycles, and thus the least common multiple of $1, \ldots, d$. Since k was chosen minimal, $k-1$ (the order of at least one element) devises the least common multiple of $1, \ldots, d$.

Let us choose $d = 2$. Then k has to be 2 or 3. Semigroups satisfying $x \approx x^2$ are called bands. This paper deals with semigroups satisfying $x \approx x^3$. In a semigroup satisfying the equation $x \approx x^2$, for all $a \in S$, there is $b \in S$ with $aba = bab = a$ [1]. Bijev and Todorov [1,2] have called such semigroups coregular. We shall characterize all coregular semigroups in the ideal $K(n, 2)$. In the next section, we give some basic definitions, notations, and facts. The third section is devoted to the main theorem and its proof. Since each idempotent semigroup is coregular, this result also describes the idempotent semigroups within $K(n, 2)$. For example, idempotent transformation semigroups are studies in [9].

2. Basic concepts. Let n be a non-negative integer and T_n be the full transformation semigroup, i.e. a set of all self-mappings from $[n] := \{1, \ldots, n\}$ into $[n]$. A subsemigroup of T_n is called a transformation semigroup. We write mappings with their argument on the left and compose from left to right. $\mathrm{Cor}(T_n)$ denotes the set of all coregular elements in T_n. It is well known [7] that the ideals of T_n are of the form $K(n, r) := \{\alpha \in T_n \mid |\mathrm{im}\alpha| \leq r\}$ for $1 \leq r \leq n$, imα denotes the image of α. Let $J(n, 2)$ be the set of all transformations with $|\mathrm{im}\alpha| = 2$. $J(n, 2)$ is a $\mathcal{J}$-class of T_n. For $\alpha \in J(n, 2)$, kerα be the kernel of α, where the factor set $[n]/_{\ker\alpha}$ is a two-element set and we write $\mathrm{im}\alpha \in \ker\beta$ ($\beta \in J(n, 2)$) in order to express that $(a, b) \in \ker\beta$ whenever $\mathrm{im}\alpha = \{a, b\}$. If α is coregular, then $\mathrm{im}\alpha \in \ker\alpha$.

As it is already mentioned, $K(n, 1) \subseteq \mathrm{Cor}(T_n)$. This paper deals with the ideal $K(n, 2)$. We investigate the maximal coregular subsemigroups of $K(n, 2)$, i.e. all coregular semigroups $S \leq K(n, 2)$ such that $\langle S, \alpha \rangle \nsubseteq \mathrm{Cor}(T_n)$ for all $\alpha \in K(n, 2) \setminus S$. Clearly, $K(n, 2)$ is not coregular for $n > 2$.

We frequently need the fact that $\mathrm{im}\alpha \notin \ker\alpha$ for any coregular transformation α in $J(n, 2)$. It shows when the product of two (coregular) transformations gives a coregular transformation in $J(n, 2)$.

Lemma 2.1. *Let* $\alpha, \beta \in \mathrm{Cor}(T_n) \cap K(n,2)$ *with* $\mathrm{im}\alpha \notin \ker\beta$. *Then* $\alpha\beta \in \mathrm{Cor}(T_n)$ *if and only if* $\mathrm{im}\beta \notin \ker\alpha$.

Proof. Since $\alpha\beta \in \mathrm{Cor}(T_n)$, we get $\mathrm{im}\alpha\beta \notin \ker\alpha\beta$. Then $\mathrm{im}\alpha \in \mathrm{im}\alpha\beta \in \ker\alpha\beta = \ker\alpha$ and conversely. □

Let $S \leq K(n,2)$ be a semigroup. Then we denote by $\mathrm{im}S := \bigcup\{\mathrm{im}\alpha \mid \alpha \in S \cap J(n,2)\}$. First, we show that there is a decomposition A_1, A_2 of $\mathrm{im}S$ ($A_1 \cup A_2 = \mathrm{im}S$ and $A_1 \cap A_2 = \varnothing$) written by $A_1 \dot\cup A_2 = \mathrm{im}S$ such that for any $\alpha \in S \cap J(n,2)$, $\mathrm{im}\alpha$ is transversal of $A_1 \dot\cup A_2$. We say that $\mathrm{im}\alpha$ is transversal of $A_1 \dot\cup A_2$ if $\mathrm{im}\alpha$ has exactly one common element with both A_1 and A_2.

Lemma 2.2. *Let* $S \leq K(n,2)$. *Then there is a decomposition* $A_1 \dot\cup A_2 = \mathrm{im}S$ *such that* $\mathrm{im}\alpha$ *is a transversal of* $A_1 \dot\cup A_2$ *for all* $\alpha \in S \cap J(n,2)$.

Proof. If there is exactly one $\alpha \in S \cap J(n,2)$, then all is clear with $\mathrm{im}\alpha = \{a,b\} = \{a\} \dot\cup \{b\}$. Suppose that $S \cap J(n,2)$ does contain $n+1$ elements and for any subset T of S with $T \cap J(n,2)$ containing n elements there is a decomposition $A_1 \dot\cup A_2 = \mathrm{im}T$ such that $\mathrm{im}\alpha$ is a transversal of $A_1 \dot\cup A_2$ for all $\alpha \in T$. If there is $\alpha \in S \cap J(n,2)$ with $\mathrm{im}\alpha \cap \mathrm{im}\beta = \varnothing$ for all $\beta \in S \setminus \{\alpha\} =: T$, then there is a decomposition $A_1 \dot\cup A_2 = \mathrm{im}T$ such that $\mathrm{im}\beta$ is a transversal of $A_1 \dot\cup A_2$ for all $\beta \in T \cap J(n,2)$. If $\mathrm{im}\alpha = \{a,b\}$, then $(A_1 \cup \{a\}) \dot\cup (A_2 \cup \{b\}) = \mathrm{im}S$ such that $\mathrm{im}\beta$ is a transversal of $(A_1 \cup \{a\}) \dot\cup (A_2 \cup \{b\})$ for all $\beta \in S \cap J(n,2)$. If there is $\alpha \in S \cap J(n,2)$ with $\mathrm{im}\alpha \cap im\beta \subseteq \{a\}$ for all $\beta \in S \setminus \{\alpha\} := T$, then there is a decomposition $A_1 \dot\cup A_2 = \mathrm{im}T$ such that $a \in A_1$ and $\mathrm{im}\beta$ is a transversal of $A_1 \dot\cup A_2$ for all $\beta \in T \cap J(n,2)$. If $\mathrm{im}\alpha = \{a,b\}$, then $A_1 \dot\cup (A_2 \cup \{b\}) = \mathrm{im}S$ such that $\mathrm{im}\beta$ is a transversal of $A_1 \dot\cup (A_2 \cup \{b\})$ for all $\beta \in S \cap J(n,2)$.

Suppose that $\mathrm{im}T \cap \mathrm{im}\alpha = \mathrm{im}\alpha$ for all $\alpha \in S \cap J(n,2)$ whenever $T = S \setminus \{\alpha\}$. Let

$$\alpha \in S \cap J(n,2) \text{ with } \mathrm{im}\alpha = \{a_1, a_2\} \text{ and } T_\alpha := S \setminus \{\alpha\}.$$

Then there is a decomposition $\mathrm{im}T_\alpha = A_1^\alpha \dot\cup A_2^\alpha$ such that $\mathrm{im}\beta$ is a transversal of $A_1^\alpha \dot\cup A_2^\alpha$ for all $\beta \in T_\alpha \cap J(n,2)$. We put

$$\alpha(a_i) := \{y \in \mathrm{im}T_\alpha \mid \text{ there is } \beta \in T_\alpha \text{ such that } \mathrm{im}\beta = \{y, a_i\}\}$$

for $i = 1,2$. If $\alpha(a_1) = \alpha(a_2) = \varnothing$, then $\mathrm{im}\beta$ is a transversal of the decomposition $(A_1^\alpha \cup \{a_1\}) \dot\cup (A_2^\alpha \cup \{a_2\})$ for all $\beta \in S \cap J(n,2)$. If $\alpha(a_1) \neq \varnothing$ or $\alpha(a_2) \neq \varnothing$, we are going to show that $\alpha(a_1) \cap \alpha(a_2) = \varnothing$. Otherwise, there exists $y \in \alpha(a_1) \cap \alpha(a_2)$ and $\beta_1, \beta_2 \in T_\alpha$ such that $\mathrm{im}\beta_i = \{y, a_i\}$ for $i = 1,2$. Without loss of generality, let $y\alpha = a_1\alpha$. We have $a_1\beta_1 = a_2\beta_1$. Indeed, $a_1\beta_1 \neq a_2\beta_1$ implies $\alpha\beta_1 \in J(n,2)$

with $\mathrm{im}\alpha\beta_1 = \mathrm{im}\beta_1 = \{y, a_1\}$ and $(y, a_1) \in \ker\alpha = \ker\alpha\beta$. Lemma 2.1 gives $\alpha\beta \notin \mathrm{Cor}(T_n)$, a contradiction. Without loss of generality, we can assume that $\alpha(a_1) \neq \varnothing$, i.e. there are $b_1 \in \alpha(a_1)$ and $\beta \in T_\alpha$ such that $\mathrm{im}\beta = \{a_1, b_1\}$. Let us put $T_\beta := S \setminus \{\beta\}$. Then there is a decomposition $\mathrm{im}T_\beta = A_1^\beta \dot\cup A_2^\beta$ such that $\mathrm{im}\gamma$ is a transversal of $A_1^\beta \dot\cup A_2^\beta$ for all $\gamma \in T_\beta \cap J(n, 2)$. Without loss of generality, we can assume that $a_1 \in A_1^\beta$ and $a_2 \in A_2^\beta$. We put

$$\beta(b_1) := \{y \in imT_\beta \mid \text{there is } \gamma \in T_\beta \text{ such that } \mathrm{im}\gamma = \{y, b_1\}\}.$$

If $\beta(b_1) \subseteq A_1^\beta$, then $\mathrm{im}\gamma$ is a transversal of the decomposition $A_1^\beta \dot\cup (A_2^\beta \cup \{b_1\})$ for all $\gamma \in S \cap J(n, 2)$. Otherwise, there is $c_1 \in \beta(b_1) \cap A_2^\beta$ and $\gamma \in T_\beta$ such that $\mathrm{im}\gamma = \{c_1, b_1\}$. By the previous considerations, $\beta(b_1) \cap \alpha(a_1) = \varnothing$. We put $T_\gamma := S \setminus \{\gamma\}$. Then there is a decomposition $\mathrm{im}T_\gamma = A_1^\gamma \dot\cup A_2^\gamma$ such that $\mathrm{im}\delta$ is a transversal of $A_1^\gamma \dot\cup A_2^\gamma$ for all $\delta \in T_\gamma \cap J(n, 2)$. Without loss of generality, we can assume that $a_1 \in A_1^\gamma$. Then $a_2 \in A_2^\gamma$, $\alpha(a_1) \subseteq A_2^\gamma$ and $\alpha(a_2) \cup \beta(b_1) \subseteq A_1^\gamma$. We put

$$\gamma(c_1) := \{y \in \mathrm{im}T_\gamma \mid \text{ there is } \delta \in T_\gamma \text{ such that } \mathrm{im}\delta = \{y, c_1\}\}.$$

If $\gamma(c_1) \subseteq A_1^\gamma$, then $\mathrm{im}\delta$ is a transversal of the decomposition $A_1^\gamma \dot\cup (A_2^\gamma \cup \{c_1\})$ for all $\delta \in S \cap J(n, 2)$. Otherwise, we continue in the same matter as before. Proceeding this way, in each step we get either a decomposition $B_1 \dot\cup B_2$ of $\mathrm{im}S$ such that $\mathrm{im}\sigma$ is a transversal of the decomposition $B_1 \dot\cup B_2$ for all $\sigma \in S \cap J(n, 2)$ or there are $\delta \in S \cap J(n, 2)$ which are not used yet in this context in any previous step and a decomposition $B_1 \dot\cup B_2$ of $\mathrm{im}T$, where $T = S \setminus \{\delta\}$ such that $\mathrm{im}\sigma$ is a transversal of the decomposition $B_1 \dot\cup B_2$ for all $\sigma \in T \cap J(n, 2)$. Since S is finite, this procedure finishes, i.e. there is a decomposition $B_1 \dot\cup B_2$ of $\mathrm{im}S$ such that $\mathrm{im}\sigma$ is a transversal of $B_1 \dot\cup B_2$ for all $\sigma \in S \cap J(n, 2)$. □

3. Main result. We show that coregular transformation semigroups in $K(n, 2)$ correspond to equivalence relations with particular properties. If S is a coregular semigroup, then there is a decomposition $A_1 \dot\cup A_2$ of $A := \mathrm{im}S$ by Lemma 2.2. We put $\mathrm{Tr}(A) := \{\{a, b\} \mid a \in A_1, b \in A_2\}$. A subset $U \subseteq \mathrm{Tr}(A)$ can be considered as a bipartite graph with $\bigcup U$ as the set of vertices and U as the set of edges. The concepts of a complete bipartite graph and the components of a disconnected graph are well known. Finally, for an equivalence relation R on U, $U/_R$ denotes the set of equivalence classes. Now we are able to state and prove the main theorem.

Theorem 3.1. *Let $S \subseteq \mathrm{Cor}(T_n) \cap K(n, 2)$. S is a maximal coregular subsemigroup of $K(n, 2)$ if and only if there are a set $A \subseteq [n]$, a decomposition $A_1 \dot\cup A_2 = A$, a subset $\mathrm{Tr} \subseteq \mathrm{Tr}(A)^2$, and an equivalence relation R on Tr such that for* *each* $\overline{a} \in \mathrm{Tr}/_R$ *the components of the graph* $(\bigcup \overline{a}, \overline{a})$ *are complete bipartite graphs and any* x *and* y *do not belong to the same component of the graph* $(\bigcup(\mathrm{Tr}\backslash\overline{a}), \mathrm{Tr}\backslash\overline{a})$ *whenever* $\{x, y\} \in \overline{a}$ *with* $S = \{\alpha \in J(n, 2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u, v) \notin \ker\alpha$ *iff* $\{u, v\} \in [\mathrm{im}\alpha]_R$ *for all* $\{u, v\} \in \mathrm{Tr}\} \cup K(n, 1)$.

Proof. Let $S = \{\alpha \in J(n, 2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u, v) \notin \ker\alpha \text{ iff } \{u, v\} \in [\mathrm{im}\alpha]_R$ for all $\{u, v\} \in \mathrm{Tr}\} \cup K(n, 1)$. Let $\{u, v\} \in \mathrm{Tr}$. Since any x and y do not belong to the same component of the graph $(\bigcup(\mathrm{Tr} \setminus [\{u, v\}]_R), \mathrm{Tr} \setminus [\{u, v\}]_R)$ whenever $\{x, y\} \in [\{u, v\}]_R$, there is a decomposition $B_1 \dot\cup B_2 = [n]$ such that $\{x, y\}$ is a transversal of $B_1 \dot\cup B_2$ if and only if $\{x, y\} \in [\{u, v\}]_R$. In particular, $\{u, v\} \in [\{u, v\}]_R$. This shows $S \subseteq \mathrm{Cor}(T_n)$ and for any $\{u, v\} \in \mathrm{Tr}$ there is a $\beta \in S$ with $\mathrm{im}\beta = \{u, v\}$.

First, we show that S is a semigroup. Let $\alpha, \beta \in S \cap J(n, 2)$ such that $\mathrm{im}\alpha \notin \ker\beta$. Then $\mathrm{im}\alpha$, $\mathrm{im}\beta \in \mathrm{Tr}$. Hence $\mathrm{im}\alpha \in [\mathrm{im}\beta]_R$, i.e. $(\mathrm{im}\alpha, \mathrm{im}\beta) \in R$. This shows $\alpha\beta \in S$ because of $\mathrm{im}\alpha\beta = \mathrm{im}\beta$ and $\ker\alpha\beta = \ker\alpha$. (Note that the remaining cases give $\alpha\beta \in I(n, 1) \subseteq S$.) Thus, S is a semigroup.

Next, we show that S is maximal. Let $\alpha \in K(n, 2) \setminus S$ be coregular. Then the following three cases are possible:

1) $\mathrm{im}\alpha \notin \mathrm{Tr}$: Let $\{u, v\} \in \mathrm{Tr}$. Suppose that $(u, v) \in \ker\alpha$. Then there is $\beta \in S$ such that $\mathrm{im}\beta = \{u, v\}$ and $(u, v) \notin \ker\beta$. Then $\alpha\beta \notin \mathrm{Cor}(T_n)$ by Lemma 2.1. Suppose that $(u, v) \notin \ker\alpha$. Then there is $\beta \in S$ such that $\mathrm{im}\beta = \{u, v\}$ but $\mathrm{im}\alpha \in \ker\beta$. Then $\mathrm{im}\beta \notin \ker\alpha$ and thus $\beta\alpha \notin \mathrm{Cor}(T_n)$ by Lemma 2.1.

2) $\mathrm{im}\alpha \in \mathrm{Tr}$ and $(u, v) \notin \ker\alpha$ for some $\{u, v\} \in \mathrm{Tr} \setminus [\mathrm{im}\alpha]_R$: Then there is $\beta \in S$ with $\mathrm{im}\beta = \{u, v\}$ and $\mathrm{im}\alpha \in \ker\beta$ since $\mathrm{im}\alpha \in \mathrm{Tr} \setminus [\mathrm{im}\beta]_R$ (R is equivalence relation and $\mathrm{im}\beta \notin [\mathrm{im}\alpha]_R$). Then $\mathrm{im}\beta \notin \ker\alpha$ implies $\beta\alpha \notin \mathrm{Cor}(T_n)$ by Lemma 2.1.

3) $\mathrm{im}\alpha \in \mathrm{Tr}$ and $(u, v) \in \ker\alpha$ for some $\{u, v\} \in [\mathrm{im}\alpha]_R$: Then there is $\beta \in S$ with $\mathrm{im}\beta = \{u, v\}$ and $\mathrm{im}\alpha \notin \ker\beta$ since $(\mathrm{im}\alpha, \mathrm{im}\beta) \in R$. Thus $\alpha\beta \notin \mathrm{Cor}(T_n)$ by Lemma 2.1.

Altogether, this shows that S is maximal.

Conversely, let S be a maximal coregular semigroup within $K(n, 2)$. Clearly, $K(n, 1) \subseteq S$. Since S is finite, there are sets $S_1, \ldots, S_m \subseteq J(n, 2)$ (for some integer m) such that $S = K(n, 1) \cup \bigcup_{i=1}^{m} S_i$ where for $1 \leq i \leq m$ it hold

i) $S_i S_i \subseteq S_i$;

ii) $\langle S_i, \alpha \rangle \nsubseteq J(n, 2)$ for all $\alpha \in S \setminus S_i$.

Clearly, $S_1, \ldots, S_m$ is a decomposition of S. Let us put

$$R := \bigcup_{i=1}^{m} \{(\mathrm{im}\alpha_1, \mathrm{im}\alpha_2) \mid \alpha_1, \alpha_2 \in S_i\}.$$

R is a relation on $\mathrm{Tr} := \{\mathrm{im}\alpha \mid \alpha \in S \cap J(n,2)\}$. By Lemma 2.2, there is a decomposition $A_1 \dot\cup A_2$ of $\mathrm{im}S$. It is easy to see that $\mathrm{Tr} \subseteq \mathrm{Tr}(\mathrm{im}S)$. First, we show that R is an equivalence relation. Let $\alpha \in S$. Then there is $i \in \{1,\ldots,m\}$ with $\alpha \in S_i$. Thus, $(\mathrm{im}\alpha, \mathrm{im}\alpha) \in R$. Hence R is reflexive. Clearly, R is symmetric. To check R is transitive, let $\alpha, \beta, \gamma, \delta \in S$ with $(\mathrm{im}\alpha, \mathrm{im}\beta) \in R$ and $(\mathrm{im}\gamma, \mathrm{im}\delta) \in R$ such that $\mathrm{im}\beta = \mathrm{im}\gamma$. Then there are $i, j \in \{1,\ldots,m\}$ with $\alpha, \beta \in S_i$ and $\gamma, \delta \in S_j$. If $i \neq j$, then there is $\varphi \in S_i$ such that $\mathrm{im}\varphi \in \ker\gamma$ but $\mathrm{im}\gamma \notin \ker\varphi$ (since $\mathrm{im}\beta = \mathrm{im}\gamma$ and $\beta \in S_i$). Thus $\gamma\varphi \notin \mathrm{Cor}(T_n)$ by Lemma 2.1, a contradiction. This shows that $i = j$ and thus $\alpha, \delta \in S_i$, i.e. $(\mathrm{im}\alpha, \mathrm{im}\delta) \in R$. Consequently, R is an equivalence relation.

Let $\alpha \in S_i$ for some $i \in \{1,\ldots,m\}$. We show that the components of the graph $(\bigcup [\mathrm{im}\alpha]_R, [\mathrm{im}\alpha]_R)$ are complete bipartite graphs. Clearly, $(\bigcup [\mathrm{im}\alpha]_R, [\mathrm{im}\alpha]_R)$ is bipartite. Suppose, there are $\{a,c\}, \{a,b\}, \{d,c\} \in [\mathrm{im}\alpha]_R$. Then there are $\beta, \gamma, \varepsilon \in S_i$ with $\mathrm{im}\beta = \{a,c\}$, $\mathrm{im}\gamma = \{d,c\}$, and $\mathrm{im}\varepsilon = \{a,b\}$. Let $\varphi \in S_i$. Then $(a,b), (a,c), (c,d) \notin \ker\varphi$. Since $[n]/_{\ker\varphi}$ has exactly two elements, this implies $(b,c) \in \ker\varphi$, $(a,d) \in \ker\varphi$ and consequently $(b,d) \notin \ker\varphi$. We consider the coregular transformation δ with $\mathrm{im}\delta = \{b,d\}$ and $\ker\delta = \ker\gamma$ and show that $\langle S, \delta\rangle \subseteq \mathrm{Cor}(T_n)$. Let $\varphi \in S$ with $\mathrm{im}\varphi \in \ker\gamma = \ker\delta$. It is clear that then $\mathrm{im}\gamma \in \ker\varphi$, $\mathrm{im}\gamma\varepsilon = \mathrm{im}\varepsilon \in \ker\varphi$, and $\mathrm{im}\gamma\beta = \mathrm{im}\beta \in \ker\varphi$. This gives $(b,d) \in \ker\varphi$ since $\ker\varphi$ is an equivalence relation. This shows, if S does not contain any φ with $\mathrm{im}\varphi = \{b,d\}$, then $\langle S, \delta\rangle \subseteq \mathrm{Cor}(T_n)$. This contradicts the maximality of S. Hence S contains some transformation $\rho \in \mathrm{Cor}(T_n)$ with $\mathrm{im}\rho = \{b,d\}$. We show $\rho \in S_i$. In fact, $\mathrm{im}\rho \notin \ker\varphi$ for all $\varphi \in S_i$ and $\rho\varphi \in S$ is coregular provides $\mathrm{im}\varphi \notin \ker\rho$ (i.e. $\rho\varphi \in J(n,2)$) for all $\varphi \in S_i$. Hence $\varphi\rho \in J(n,2)$ for all $\varphi \in S_i$. This shows that $\rho \in S_i$. Consequently, $\{b,d\} \in [\mathrm{im}\alpha]_R$.

We have to show that any x and y do not belong to the same component of the graph $(\bigcup(\mathrm{Tr} \setminus [\mathrm{im}\alpha]_R), \mathrm{Tr} \setminus [\mathrm{im}\alpha]_R)$ whenever $\{x,y\} \in [\mathrm{im}\alpha]_R$ for all $\alpha \in S$. Let $\alpha \in S_i$ for some $i \in \{1,\ldots,m\}$ and let $\mathrm{im}\beta \in \mathrm{Tr} \setminus [\mathrm{im}\alpha]_R$ for a suitable $\beta \in S \setminus S_i$. Assume that $\mathrm{im}\beta \notin \ker\rho$ for some $\rho \in S_i$. Then $\beta\rho\varphi \in S \cap J(n,2)$ for all $\varphi \in S_i$. Then $\mathrm{im}\varphi \notin \ker\beta$ for all $\varphi \in S_i$ since S is coregular. Assume that there is $\delta \in S_i$ with $\mathrm{im}\beta \in \ker\delta$. In particular, $\mathrm{im}\delta \notin \ker\beta$ and thus $\delta\beta \notin \mathrm{Cor}(T_n)$ by Lemma 2.1, a contradiction. So, $\mathrm{im}\beta \notin \ker\varphi$ for all $\varphi \in S_i$. Altogether, $\langle S_i, \beta\rangle \subseteq J(n,2)$ is a contradiction. Hence $\mathrm{im}\beta \in \ker\varphi$ for all $\varphi \in S_i$. Since $\ker\alpha$ is an equivalence relation, this gives the argument that the set of edges of a component of the graph $(\bigcup(\mathrm{Tr} \setminus [\mathrm{im}\alpha]_R), \mathrm{Tr} \setminus [\mathrm{im}\alpha]_R)$ lies in one equivalence class of $\ker\varphi$ for any $\varphi \in S_i$, i.e. any x and y do not belong to the same component of the graph $(\bigcup(\mathrm{Tr} \setminus [\mathrm{im}\alpha]_R), \mathrm{Tr} \setminus [\mathrm{im}\alpha]_R)$ whenever $\{x,y\} \in [\mathrm{im}\alpha]_R$.

It remains to show that $S = \{\alpha \in J(n,2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u,v) \notin \ker\alpha$ iff $\{u,v\} \in [\mathrm{im}\alpha]_R$ for all $\{u,v\} \in \mathrm{Tr}\} \cup K(n,1)$. Let $\varphi \in S \cap J(n,2)$. Then $\varphi \in S_i$ for some $i \in \{1,\ldots,m\}$, $\mathrm{im}\varphi \in \mathrm{Tr}$ and $(u,v) \notin \ker\varphi$ for all $\{u,v\} \in [\mathrm{im}\varphi]_R$ since $S_iS_i \subseteq J(n,2)$. If $\{u,v\} \in \mathrm{Tr} \setminus [\mathrm{im}\varphi]_R$, then we have already shown that

$\{u,v\} \in \ker\varphi$. Hence $\varphi \in \{\alpha \in J(n,2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u,v) \notin \ker\alpha$ iff $\{u,v\} \in [\mathrm{im}\alpha]_R$ for all $\{u,v\} \in \mathrm{Tr}\}$. Thus $S \subseteq \{\alpha \in J(n,2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u,v) \notin \ker\alpha$ iff $\{u,v\} \in [\mathrm{im}\alpha]_R$ for all $\{u,v\} \in \mathrm{Tr}\} \cup K(n,1)$. In the first part, we have already proved that $\{\alpha \in J(n,2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u,v) \notin \ker\alpha$ iff $\{u,v\} \in [\mathrm{im}\alpha]_R$ for all $\{u,v\} \in \mathrm{Tr}\} \cup K(n,1)$ is a coregular semigroup. The maximality of S implies $S = \{\alpha \in J(n,2) \mid \mathrm{im}\alpha \in \mathrm{Tr}, (u,v) \notin \ker\alpha$ iff $\{u,v\} \in [\mathrm{im}\alpha]_R$ for all $\{u,v\} \in \mathrm{Tr}\} \cup K(n,1)$. □

We finish with the construction of a maximal coregular semigroup (as an example) using Theorem 3.1.

Example 3.1. If $n = 3$ and $\{1,2,3\} = \{1,2\} \dot\cup \{3\}$, then it is easy to verify that the relation

$$R := \{(\{1,3\},\{1,3\}),(\{1,3\},\{2,3\}),(\{2,3\},\{1,3\}),(\{2,3\},\{2,3\})\}$$

satisfies all conditions of the relation R explained in Theorem 3.1. Hence R corresponds to a coregular subsemigroup of $K(3,2)$ by Theorem 3.1. It is easy to calculate that the semigroup S defined in Theorem 3.1 is exactly the following:

$$S = \left\{ \begin{pmatrix} \overline{123} \\ 1 \end{pmatrix}, \begin{pmatrix} \overline{123} \\ 2 \end{pmatrix}, \begin{pmatrix} \overline{123} \\ 3 \end{pmatrix}, \begin{pmatrix} \overline{12} & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \overline{12} & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} \overline{12} & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} \overline{12} & 3 \\ 3 & 2 \end{pmatrix} \right\}.$$

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