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MATHEMATIQUES
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EXPLICIT SOLUTION OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR HEAT EQUATION

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Abstract

An initial-boundary value problem for one-dimensional heat equation with a nonlocal boundary condition is studied. This boundary condition ensures both simple and triple eigenvalues of the corresponding one-dimensional spectral problem. Applying spectral projectors, we find a series solution of the problem for a special choice of the initial function. Then, using operational calculus approach of Dimovski, we obtain an explicit representation of the solution in the general case. The expression obtained contains a non-classical convolution product of the particular solution and an arbitrary initial function. This result is an extension of the classical Duhamel principle, but for the space variable.

Key words: nonlocal spectral problem, spectral projector, nonclassical convolution, operational calculus, Duhamel principle

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Here a nonlocal boundary value problem for the classical heat equation with a two-point boundary functional is considered. A special feature of the problem is the presence both of single and triple eigenvalues of the corresponding one-dimensional spectral problem. For example, in Ionkin [1] all the eigenvalues are with double multiplicity. In [2], cases of simple and double eigenvalues are considered.

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Here we propose for the first time explicit representation in a case both with simple and triple eigenvalues.

Let us consider the following initial-boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = F(x,t), \quad 0 < x < 1, 0 < t, \tag{1}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = 0, \quad 0 \le t, \tag{3}$$

$$2u\left(\frac{1}{2},t\right) + u(1,t) = 0, \quad 0 \le t. \tag{4}$$

The last boundary condition is a nonlocal one of Bitsadze–Samarskii type. The original Bitsadze–Samarskii [3] condition would look as: $u\left(\frac{1}{2},t\right) - u(1,t) = 0$. It is introduced in [3] and studied in [4].

1. A spectral problem, connected with BVP (1)–(4). In order to find a series solution of (1)–(4), we are to consider the following nonlocal eigenvalue problem in $C^2([0,1])$:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad 2y\left(\frac{1}{2}\right) + y(1) = 0. \tag{5}$$

A special feature of the problem is the presence of triple eigenvalues along with simple ones. Indeed, as it is easy to see, each eigenvalue $-\lambda_n^2$ of (5) could be obtained from a zero λ_n of the entire function

$$E(\lambda) = \frac{1}{2}\left(\frac{\sin\frac{\lambda}{2}}{\lambda} + \frac{\sin\lambda}{\lambda}\right) = \frac{2}{\lambda}\sin\frac{\lambda}{4}\cos^3\frac{\lambda}{4}. \tag{6}$$

The zeros of $E(\lambda)$ determined by $\frac{1}{\lambda}\sin\frac{\lambda}{4} = 0$ are $\mu = 4k\pi$, $k \in \mathbb{Z}\setminus 0$ and they are simple ones. The corresponding eigenvalues are $-\mu_k^2 = -(4k\pi)^2$, $k \in \mathbb{N}$. (Here we take only a half of the zeros $\mu_k = 4k\pi$, $k \in \mathbb{Z}\setminus 0$.) Denote by $\mathcal{E}_{\mu_k}$ the one-dimensional linear space spanned on the eigenfunction $\sin\mu_k x$, $k = 1,2,\ldots$.

The equation $\cos^3\frac{\lambda}{4} = 0$ determines the triple zeros $\lambda_n = 2(2n-1)\pi$, $n \in \mathbb{Z}$ of $E(\lambda)$. The corresponding eigenvalues are $-\lambda_n^2 = -4(2n-1)^2\pi^2$, $n \in \mathbb{N}$. Denote by $\mathcal{E}_{\lambda_n}$, $n = 1,2,\ldots$ the three-dimensional linear space spanned on the eigenfunction $\sin\lambda_n x$ and associated eigenfunctions $x\cos\lambda_n x$ and $x^2\sin\lambda_n x$.

The resolvent operator $R_{-\lambda^2}$ of the spectral problem (5) is determined by the solution $y = R_{-\lambda^2}f(x)$ of boundary value problem

$$\frac{d^2y}{dx^2} + \lambda^2 y = f(x), \quad 0 < x < 1, \quad y(0) = 0, \quad 2y\left(\frac{1}{2}\right) + y(1) = 0. \tag{7}$$

Explicitly,

$$R_{-\lambda^2}f(x) = \frac{1}{\lambda}\int_0^x f(\xi)\sin\lambda(x-\xi)d\xi$$
$$-\left(\int_0^{\frac{1}{2}} f(\eta)\sin\lambda(\frac{1}{2}-\eta)d\eta + \frac{1}{2}\int_0^1 f(\eta)\sin\lambda(1-\eta)d\eta\right)\frac{\sin\lambda x}{\lambda^2 E(\lambda)}.$$

$R_{-\lambda^2}f(x)$ is determined for all $\lambda \in \mathbb{C}$, except for the zeros of $E(\lambda)$. The resolvent operator $R_{-\lambda^2}$ is defined for $\lambda = 0$ since $E(0) = 1$. Further, for the aims of our operational calculus, it is convenient to use only the operator R_0. Then, denoting $L_x f(x) = R_0 f(x)$, we have

$$L_x f(x) = \int_0^x (x-\xi)f(\xi)d\xi - x\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\eta\right)f(\eta)d\eta + \frac{1}{2}\int_0^1 (1-\eta)f(\eta)d\eta\right).$$

L_x is the right inverse operator of $\dfrac{d^2}{dx^2}$ determined by the boundary conditions $(L_x f)(0) = 0$, $2(L_x f)(\frac{1}{2}) + L_x f(1) = 0$.

2. Spectral projectors. ***a) For simple eigenvalues.*** Let us consider the eigenvalues $-\mu_k^2$ with $\mu_k = 4k\pi$, $k = 1, 2, \ldots$. Then the spectral Riesz' projectors $P_{\mu_k} : C([0,1]) \to \mathcal{E}_{\mu_k} = \text{span}\{\sin\mu_k x\}$ are

(8)
$$P_{\mu_k}\{f\} = \frac{1}{\pi i}\int_{\Gamma_{\mu_k}} R_{-\mu^2}f(x)\mu d\mu$$
$$= \left(2\int_0^{\frac{1}{2}} f(\xi)\sin\mu_k\xi d\xi + \int_0^1 f(\xi)\sin\mu_k\xi d\xi\right)\sin\mu_k x,$$

where Γ_{μ_k} is a simple contour containing the zero μ_k only.

b) For triple eigenvalues. Let us consider the eigenvalues $-\lambda_n^2$ with $\lambda_n = 2(2n-1)\pi$, $n = 1, 2, \ldots$. Then the corresponding spectral Riesz' projectors

$$Q_{\lambda_n} : C([0,1]) \to \mathcal{E}_{\lambda_n} = \text{span}\{\sin\lambda_n x,\ x\cos\lambda_n x,\ x^2\sin\lambda_n x\}$$

are

$$
\begin{aligned}
(9) \qquad Q_{\lambda_n}\{f\} &= \frac{1}{\pi i}\int_{\Gamma_{\lambda_n}} R_{-\lambda^2} f(x)\lambda d\lambda \\
&= \left(\int_0^1 (8\xi^2-16\xi+7)f(\xi)\sin\lambda_n\xi d\xi \right.\\
&\left. -2\int_0^{\frac{1}{2}}(8\xi^2-8\xi+1)f(\xi)\sin\lambda_n\xi d\xi\right)\sin\lambda_n x\\
&+\left(\int_0^1(1-\xi)f(\xi)\cos\lambda_n\xi d\xi-\int_0^{\frac{1}{2}}(1-2\xi)f(\xi)\cos\lambda_n\xi d\xi\right)16x\cos\lambda_n x\\
&+\left(\int_0^1 f(\xi)\sin\lambda_n\xi d\xi-2\int_0^{\frac{1}{2}}f(\xi)\sin\lambda_n\xi d\xi\right)8x^2\sin\lambda_n x.
\end{aligned}
$$

Definition 1. *Let $f \in C([0,1])$. The formal spectral expansion of $f(x)$ for eigenvalue problem* (5) *is the correspondence*

$$f(x) \sim \sum_{k=1}^{\infty}\left(P_{\mu_k}+Q_{\lambda_k}\right). \tag{10}$$

In fact, this formal spectral expansion, is not completely formal, since it has a uniqueness property if $P_{\mu_k}\{f\}=0$ for $k=1,2,\dots$ and $Q_{\lambda_k}\{f\}=0$ for $k=1,2,\dots$, then $f\equiv 0$. This follows immediately from a theorem of Bozhinov (see [5]). In general, it is not supposed the series in (10) to be convergent. If additionally it happens this series to be uniformly convergent on $[0,1]$, then $f(x)=\sum_{k=1}^{\infty}(P_{\mu_k}+Q_{\lambda_k})$. A sufficient condition for uniform convergence of the series is: $f\in C^2([0,1])$ with $f(0)=0$ and $2f\left(\frac{1}{2}\right)+f(1)=0$. Indeed, then, after integrating by parts in (8) and (9) one obtains $|P_{\mu_k}\{f\}|\le \frac{A}{k^2}$, $|Q_{\lambda_n}\{f\}|\le\frac{B}{n^2}$, where A, B are constants nondepending on k and n. This ensures uniform convergence of the spectral expansion (10) of f and thus its sum to be a function from $C([0,1])$. Applying the uniqueness property, it follows that the sum of the series (10) is exactly $f(x)$, since the spectral projectors of the right-hand side in (10) coincide with the corresponding projectors of $f(x)$.

3. Convolutions. ***The Duhamel convolution.*** The operation,

$$(\varphi \overset{t}{*} \psi)(t)=\int_0^t\varphi(t-\tau)\psi(\tau)d\tau,\quad \varphi,\psi\in C([0,\infty)), \tag{11}$$

bears the name of Duhamel, but sometimes it is called also either Borel, or Laplace convolution. It is connected with the Voltera integration operator

$$l_t\varphi(t) = \int_0^t \varphi(\tau)d\tau, \tag{12}$$

since $l_t\varphi = \{1\} \overset{t}{*} \varphi$. To say it differently, l_t is the convolution operator $\{1\} \overset{t}{*}$, i.e. $l_t = \{1\} \overset{t}{*}$.

A non-classical convolution.

Theorem 1 (Dimovski [6], p.119)). *Let us denote* $\Phi\{f(\xi)\} = f\left(\frac{1}{2}\right) + \frac{1}{2}f(1)$ *and*

$$\tilde{\Phi}_\xi\{f\} = \Phi_\xi \circ l_\xi\{f\} = \int_0^{\frac{1}{2}} f(\xi)d\xi + \frac{1}{2}\int_0^1 f(\xi)d\xi.$$

Then

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\tilde{\Phi}_\eta\{h(x,\eta)\} = -\frac{1}{2}\int_0^{\frac{1}{2}} h(x,\eta)d\eta - \frac{1}{4}\int_0^1 h(x,\eta)d\eta \tag{13}$$

with

$$h(x,\eta) = \int_x^\eta f(x+\eta-\xi)g(\xi)d\xi - \int_{-x}^\eta f(|\eta - x - \xi)g(|\xi|)sgn(\xi(\eta - x - \xi))d\xi$$

is a bilinear, commutative and associative operations in $C([0,1])$*, such that*

$$R_{-\lambda^2}\{f\} = \left\{\frac{\sin \lambda x}{\lambda E(\lambda)}\right\} \overset{x}{*} f, \tag{14}$$

with $E(\lambda) = \Phi_\xi\left\{\frac{\sin \lambda x}{\lambda}\right\} = \frac{1}{\lambda}\sin\frac{\lambda}{2} + \frac{\sin\lambda}{2\lambda}$.

For $\lambda = 0$, we get

$$L_x f = \{x\} \overset{x}{*} f. \tag{15}$$

Inserting (14) in (8), we find a useful representation

$$P_{\mu_k}\{f\} = \psi_k \overset{x}{*} f, \quad \text{with} \quad \psi_k(x) = 2\mu_k \sin 2\mu_k x. \tag{16}$$

Analogically, inserting (14) in (9) we find

$$\begin{aligned} &Q_{\lambda_n}\{f\} = \varphi_n \overset{x}{*} f, \quad \text{with} \\ &\varphi_n(x) = -2\lambda_n \sin \lambda_n x - 32x\cos\lambda_n x + 16\lambda_n x^2 \sin\lambda_n x. \end{aligned} \tag{17}$$

In [6] the corresponding theorem is stated for an arbitrary linear functional Φ in $C^1([0,1])$. Next we will combine both the Duhamel convolution (11) and the Dimovski convolution (13) into a two-dimensional convolution in $C(D) = C([0,1] \times [0,\infty))$.

Theorem 2. *Let $u, v \in C(D)$. Then the operation*

$$(u * v)(x,t) = \int_0^t u(x, t-\tau) \overset{x}{*} v(x,\tau) d\tau,$$

where $\overset{x}{*}$ is operation (13)*, is a bilinear, commutative and associative operation in $C(D)$, such that*

$$l_t L_x u(x,t) = \{x\} * u(x,t). \tag{18}$$

For a proof, see [7], where a more general case is considered.

4. The ring of multiplier fractions of $(C(D), *)$**.** We consider the convolution algebra $(C, *)$, where $C = C(D)$. Our direct operational calculus approach is based on the notion of a multiplier of the convolution algebra $(C(D), *)$ (see Larsen [8]).

Definition 2 (Larsen [8])**.** *An operator $M : C(D) \to C(D)$ is said to be a multiplier of the convolution algebra $(C(D), *)$ iff $M(f * g) = (Mf) * g$ for all $f, g \in C(D)$.*

Here we will remind only some specific notations.

The multipliers of the form $\{u(x,t)\}*$ will be denoted by $\{u\}$ or simply u and the result of the application of the operator $u*$ to a function $F \in C(D)$ will be denoted by $\{u\}F$ or uF.

Lemma 1. *Let f be a function from $C([0,1])$. The convolution operator $f \overset{x}{*}$, defined in $C(D)$ by $(f \overset{x}{*})u = f \overset{x}{*} u$ is a multiplier of the convolution algebra $(C(D), *)$.*

Lemma 2. *Let φ be a function from $C([0,\infty))$. The convolution operator $\varphi \overset{t}{*}$, defined in $C(D)$ by $(\varphi \overset{t}{*})u = \varphi \overset{t}{*} u$, is a multiplier of the convolution algebra $(C(D), *)$.*

For proofs in a more general situation, see [7].

Definition 3. *Let $f \in C([0,1])$ and $\varphi \in C([0,\infty))$, are both considered as functions of $C(D)$. The operator $[f]_x$ defined by $[f]_x u = f \overset{x}{*} u$ is said to be a* partial numerical operator *with respect to t and the operator $[\varphi]_t u = \varphi \overset{t}{*} u$ is said to be a partial numerical operator with respect to x.*

In these notations we have $L_x = [x]_t$ and $l_t = [1]_x$.

The notion of numerical operator for Duhamel convolution is introduced by Mikusiński in [9].

Lemma 3 (Larsen [8])**.** *The set of all multipliers of the convolution algebra $(C, *)$ is a commutative ring $\mathfrak{M}$.*

The multiplicative set $\mathfrak{N}$ of the non-zero non-divisors of 0 in $\mathfrak{M}$ is non-empty, since at least the operators $\{x\} \overset{x}{*} = [x]_t$ and $\{1\} \overset{t}{*} = [1]_x$ are nondivisors of 0.

Next we introduce the ring $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$ of the multiplier fractions of the form $\dfrac{A}{B}$ where $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$. The standard algebraic procedure of constructing of this ring, named "localization", is described, e.g. in Lang [10]. Basic for our construction are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $s_t = \dfrac{1}{l_t}$ in $\mathcal{M}$, of the multipliers L_x and l_t correspondingly. If $u \in C^2(D)$, then u_{xx} and u_t are connected with $S_x u$ and $s_t u$ but, in general, they are different from them.

Lemma 4. *Let $u \in C(D)$ be continuous partial derivatives u_{xx}, u_t on D. Then*

$$u_{xx} = S_x u + S_x\{(x\Phi\{1\} - 1)u(0,t)\} - \Phi_\xi\{u(\xi,t)\}, \tag{19}$$

$$u_t = s_t - [u(x,0)]_t. \tag{20}$$

Proof. Relation (20) is similar to a corresponding relation in Mikusiński [9]. Let us prove (19). It is easy to verify the identity

$$L_x\{u_{xx}\} = u(x,t) + (x\Phi\{1\} - 1)u(0,t) - x\Phi_\xi\{u(\xi,t)\}.$$

It remains to multiply it by S_x and to use $S_x\{x\} = S_x L_x = 1$, in order to get (19). □

5. Algebraization of boundary value problem (1)–(4). Relations (19) and (20) allow us to reduce both equation (1) and BVCs (2)–(4) to a single linear algebraic equation $\mathcal{M}$ for u. Indeed, substituting u_t and u_{xx} from ((19) and (20) in the equation $u_t(x,t) - u_{xx}(x,t) = F(x,t)$, we get

$$s_t u - [u(x,0)]_t - S_x u - S_x\{(x\Phi\{1\} - 1)u(0,t)\} + \Phi_{xi}\{u(\xi,t)\} = F(x,t).$$

Now using initial condition (2) and boundary value conditions (3) and (4), we obtain

$$(s_t - S_x)u = \{F(x,t)\} + [f(x)]_t. \tag{21}$$

Thus we reduced BVP (1)–(4) to the single linear algebraic equation (21) for u in $\mathcal{M}$. It is reasonable to introduce the notation of a *weak solution* of BVP (1)–(4), along with the classical solution.

Definition 4. *A function $u \in C(D)$ is said to be a weak solution of BVP* (1)–(4) *if it is a solution of* (21).

Next, let us consider the problem of uniqueness of the solution of (1) - (4). Equation (21) reduces it to the algebraic question, whether $s_t - S_x$ is a divisor of zero in $\mathcal{M}$ or not.

Theorem 3. *The element $s_t - S_x$ is a nondivisor of zero in $\mathcal{M}$.*

Proof. Assume the contrary, i.e. that there exists a nonzero multiplier fraction $\dfrac{A}{B} \neq 0$ with $(s_t - S_x)\dfrac{A}{B} = 0$. The last relation is equivalent to $(s_t - S_x)A = 0$. Since $A \neq 0$, then there exists a function $v \in C(D)$ such that $Av = u \neq 0$. Then

$(s_t - S_x)A = 0$ implies $(s_t - S_x)u = 0$. We multiply the last equation by l_t and obtain $u - S_x l_t u = 0$. If $t = 0$, then $u(x,0) = 0$. Next we multiply $(s_t - S_x)u = 0$ by $\psi_k(x)$, $k = 1, 2, \ldots$ and obtain $(s_t - S_x)v_k = 0$, where $v_k(x,t) = \alpha_k(t) \sin \mu_k x$, $k = 1, 2, \ldots$. We find

$$s_t\{\alpha_k(t) \sin \mu_k x\} - S_x\{\alpha_k(t) \sin \mu_k x\} = 0.$$

Using (19) and (20) we find

$$\frac{d\alpha_k(t)}{dt} \sin \mu_k x + \mu_k^2 \alpha_k(t) \sin \mu_k x + \alpha_k(t) \Phi_{xi}\{\sin \mu_k \xi\} = 0.$$

But $\Phi_\xi\{\sin \mu_k \xi\} = 0$ and we get

$$\frac{d\alpha_k(t)}{dt} + \mu_k^2 \alpha_k(t) = 0, \quad \alpha_k(0) = 0.$$

This initial value problem has the unique solution $\alpha_k(t) = 0$.

Let us multiply $(s_t - S_x)u = 0$ by $\varphi_n(x)$, $n = 1, 2, \ldots$. We obtain $(s_t - S_x)u_n = 0$, where $u_n(x,t) = A_n(t) \sin \lambda_n x + B_n(t) \cos \lambda_n x + C_n(t) x^2 \sin \lambda_n x$, $n = 1, 2, \ldots$. From the linear independence of $\sin \lambda_n x$, $x \cos \lambda_n x$ and $x^2 \sin \lambda_n x$ it follows:

$$\begin{aligned} &\frac{dA_n(t)}{dt} + \lambda_n^2 A_n(t) + 2\lambda_n B_n(t) - 2C_n(t) = 0, \\ &\frac{dB_n(t)}{dt} + \lambda_n^2 B_n(t) - 4\lambda_n C_n(t) = 0, \\ &\frac{dC_n(t)}{dt} + \lambda_n^2 C_n(t) = 0. \end{aligned} \tag{22}$$

From $u(x,0) = 0$ it follows $u_n(x,0) = 0$ and thus we obtain the initial values of the $A_n(t)$,$B_n(t)$ and $C_n(t)$

$$A_n(0) = 0, \quad B_n(0) = 0, \quad C_n(0) = 0. \tag{23}$$

This system (22) with the initial values (23) has the unique solution $A_n(t) = 0$, $B_n(t) = 0$, $C_n(t) = 0$. We find $u(x,t) = 0$. □

The solution of (21) in $\mathcal{M}$ is

$$u = \frac{1}{s_t - S_x}[f(x)]_t + \frac{1}{s_t - S_x}\{F(x,t)\}. \tag{24}$$

We may call (24) the *formal* (*or generalized*) *solution* of problem (1)–(4).

6. Interpretation of the formal (generalized) solution of (1)–(4) as a function. Our next task is to interpret (if possible) (24) as a function of $C(D)$. To this end, we consider a special case of problem (1)–(4) for $F(x,t) \equiv$

0 and $f(x) = L_x\{x\} = \dfrac{1}{S_x^2} = \dfrac{x^3}{6} - \dfrac{5x}{48}$. We denote its solution, if it exists, by $\Omega = \Omega(x,t)$. Having in mind that $L_x\{x\} = \dfrac{1}{S_x^2}$, we have the following algebraic representation of this solution:

$$\Omega = \frac{1}{s_t - S_x}\left[\frac{x^3}{6} - \frac{5x}{48}\right]_t = \frac{1}{S_x^2(s_t - S_x)}. \tag{25}$$

As for the special solution $\Omega(x,t)$, it can be found in an explicit series form, using the spectral projectors (8) and (9). Thus we obtain

Lemma 5. *If* $f(x) = \dfrac{x^3}{6} - \dfrac{5x}{48}$ *and* $F(x,t) \equiv 0$*, then the solution* $\Omega(x,t)$ *of the BVP* (1)–(4) *is*

$$\Omega(x,t) = \sum_{n=1}^{\infty} \Omega_n(x,t), \tag{26}$$

where

$$\Omega_n(x,t) = \frac{2e^{-\mu_n^2 t}}{\mu_n^3}\sin\mu_n x - \frac{e^{-\lambda_n^2 t}(96 + 32\lambda_n^4\, t^2 + \lambda_n^2(1+80t))}{2\lambda_n^5}\sin\lambda_n x$$
$$+\frac{e^{-\lambda_n^2 t}8(3 - 2\lambda_n^2 t)}{\lambda_n^2}\, x\cos\lambda_n x + \frac{16e^{-\lambda_n^2 t}}{\lambda_n^3}\, x^2\sin\lambda_n x,$$

$n = 1, 2, \ldots$.

The proof may be accomplished by a direct check, too.

The generalized solution of problem (1)–(4) for arbitrary $f(x)$ and $F(x,t)$ can be written in the form

$$\begin{aligned} u &= \frac{1}{s_t - S_x}[f(x)]_t + \frac{1}{s_t - S_x}F(x,t) \\ &= S_x^2\left(\frac{1}{S_x^2(s_t - S_x)}[f(x)]_t + \frac{1}{S_x^2(s_t - S_x)}F(x,t)\right). \end{aligned}$$

Under corresponding assumptions for smoothness of the functions $f(x)$ and $F(x,t)$, it can be written as a function of the form

$$u = \frac{\partial^4}{\partial x^4}\left[\Omega \overset{x}{*} f(x) + \Omega * F(x,t)\right]. \tag{27}$$

Then (27) gives the following Duhamel-type representation of the solution of (1)–(4).

Theorem 4. *Let us* $f(x) \in C^2([0,1])$, $f(0) = 0$ *and* $2f\left(\dfrac{1}{2}\right) + f(1) = 0$. *Then*

$$u = \frac{\partial^4}{\partial x^4}(\Omega(x,t) \overset{x}{*} f(x)) =$$

$$-\frac{1}{4}\left\{ \int_0^1 (\Omega_x(x+1-\xi,t) - \Omega_x(1-x-\xi,t)) f''(\xi) d\xi \right.$$

$$\left. +2\int_0^{\frac{1}{2}} (\Omega_x(x+\frac{1}{2}-\xi,t) - \Omega_x(\frac{1}{2}-x-\xi,t)) f''(\xi) d\xi \right\}$$

is a weak solution of (1)–(4) *for* $F(x,t) \equiv 0$.

The proof may be accomplished by a direct check, too.

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