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DISCUSSION ON EXP-FUNCTION METHOD AND MODIFIED METHOD OF SIMPLEST EQUATION

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Abstract

We discuss the relation between the modified method of simplest equation and the exp-function method. First, on the basis of our experience from the application of the method of simplest equation we generalize the exp-function approach. Then we apply the method for obtaining exact solutions for members of a class of nonlinear PDEs which contains as particular cases several nonlinear PDEs that model the propagation of water waves.

Key words: nonlinear partial differential equations, method of simplest equation, exact travelling-wave solutions, exp-function method

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1. Nonlinear PDEs and method of simplest equation. Nonlinear models are much used in various branches of science [1–3]. Often such models contain nonlinear PDEs and because of this the interest in obtaining exact analytical solutions of nonlinear PDEs increses steadily. Such exact solutions often describe important classes of waves and processes in the investigated systems. In addition, the exact solutions can be useful as initial conditions in the process of obtaining of numerical solutions or as test solutions for computer programmes for obtaining numerical solutions of the studied nonlinear PDEs.

Because of all above, the nonlinear PDEs are widely applied in the theory of solitons [4, 5], hydrodynamics and the theory of turbulence [6–10]; the theory of dynamical systems, chaos [11–14], etc. Sophisticated methods for obtaining exact solutions of nonlinear PDEs such as the inverse scattering transform or the method of Hirota [15] allow obtaining of soliton solutions of some equations. In the last several years effective approaches for obtaining exact special solutions of complicated nonlinear nonintegrable PDEs have been developed too [16–18]. These approaches led to exact solutions of many equations such as the Kuramoto-Shivasinsky equation [19] or equations connected to the models of migration of populations [20–22]. The discussion below will be devoted to the modified method of simplest equation: a version of the method of simplest equation for obtaining exact solutions of nonlinear PDEs and on the relation of this method to another popular method: the exp-function method.

A brief description of the method of simplest equation is as follows [23]. Let us have a partial differential equation and let by means of an appropriate presentation this equation be reduced to the nonlinear ordinary differential equation

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots\right) = 0. \tag{1}$$

For a large class of equations from kind (1) exact solution can be constructed as finite series

$$F(\xi) = \sum_{\mu=0}^{\nu} p_\mu [\Phi(\xi)]^\mu, \tag{2}$$

where $\nu > 0$, μ, p_μ are parameters and $\Phi(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation. The simplest equation is of lower order than (1) and we know the general solution of the simplest equation or we know at least exact analytical particular solution(s) of the simplest equation.

The application of the modified method of simplest equation is as follows. First, by means of an appropriate presentation (for example, the travelling-wave approach) the solved class of nonlinear PDEs is reduced to a class of nonlinear ODEs of kind (1). In the method of simplest equation the resulting ODEs are treated as in the first step of the test for Painleve property: the corresponding equation is a subject of leading order analysis that brings to determination of ν from Eq.(2). In the modified method of simplest equation one uses the equivalent procedure of obtaining and solving a balance equation as follows. First, the finite-series solution (2) is substituted in (1) and as a result a polynomial of $\Phi(\xi)$ is obtained. Equation (2) is a solution of (1) if all coefficients of the obtained polynomial of $\Phi(\xi)$ are equal to 0. Then by means of a balance equation one ensures that there are at least two terms in the coefficient of the largest power of $\Phi(\xi)$. The balance equation gives a relationship between the parameters of the solved class of equations and the parameters of the solution. The application of the balance equation and setting the coefficients of the polynomial of $\Phi(\xi)$ to 0 leads to a system of nonlinear relationships among the parameters of the solution and the parameters of the solved class of equations. Each solution of the obtained system of nonlinear algebraic equations leads to a solution of a nonlinear PDE from the investigated class of nonlinear PDEs.

2. The exp-function method: one possible generalization and application. Let us now consider the exp-function method. The standard exp-function approach for a solution of a nonlinear partial differential equation is [24]

$$u(x,t) = \frac{\sum_{i=0}^{m} a_i \exp(i\xi)}{\sum_{j=0}^{n} b_j \exp(j\xi)}, \quad \xi = kx + wt + \delta. \tag{3}$$

The authors of the exp-function method do not define the class of equations for which an exact solution can be obtained by this method. In our opinion the method can be applied to some nonintegrable partial differential equations with polynomial nonlinearity. Equation (3) can be generalized on the basis of the following observation. In one of the variants of the modified method of simplest equation the one-wave solution of the studied nonlinear partial differential equation is searched by the following kind:

$$u(\xi) = \sum_{l=0}^{L} A_l [F(\xi)]^{B_l}, \tag{4}$$

where $F(\xi)$ is a solution of the simplest equation, A_l and B_l are parameters (if $B_l = l$ Eq. (4) is a polynomial approach but B_l can be non-integer number too), and $\xi = x - vt + \xi_0$ where v is the velocity of the wave and ξ_0 is a parameter. When the equation of Bernoulli

$$\frac{dF}{d\xi} = aF(\xi) + b[F(\xi)]^M, \quad M = 2, 3, \ldots, \tag{5}$$

is used as a simplest equation its solutions are

$$F(\xi) = \left\{ \frac{a \exp[a(M-1)(\xi + \xi_0)]}{1 - b \exp[a(M-1)(\xi + \xi_0)]} \right\}^{\frac{1}{M-1}}, \quad \text{case } a > 0, b < 0,$$

$$F(\xi) = \left\{ -\frac{a \exp[a(M-1)(\xi + \xi_0)]}{1 + b \exp[a(M-1)(\xi + \xi_0)]} \right\}^{\frac{1}{M-1}}, \quad \text{case } a < 0, b > 0. \tag{6}$$

We observe that the term in $\{\ldots\}$ from Eqs (6) can be easily obtained from Eq. (3) when $n = m = 1$.

On the basis of all above the following simple generalization of the exp-function approach is obtained. One searches for an exact solution of the studied nonlinear PDE on the basis of the presentation

$$u(x,t) = \sum_{l=0}^{L} A_l \left[\frac{\sum_{i=0}^{m} a_i \exp(i\xi)}{\sum_{j=0}^{n} b_j \exp(j\xi)} \right]^{B_l}, \quad \xi = kx + wt + \delta. \tag{7}$$

Let us now apply the above approach to the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial}{\partial x}\left(\sum_{h=1}^{H} \alpha_h u^h \right) - \nu \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \gamma u \frac{\partial^3 u}{\partial x^3} = 0. \tag{8}$$

For example, particular cases of Eq. (8) are:

(I) The Camassa–Holm equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + 3u\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^3 u}{\partial x^3} = 0$$

that describes the propagation of shallow water waves over a flat bottom;

(II) The Degasperis–Processi equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + 4u\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^3 u}{\partial x^3} = 0$$

that is also connected to the dynamics of the nonlinear shallow water waves;

(III) The Fornberg–Whitham equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^3 u}{\partial x^3} = 0$$

that is used as a model for investigation of the wave-breaking. We shall search for solutions of the kind (7) with $B_l = lB$ where B is a parameter. We shall discuss the simplest possible case $n = m = 1$. Firstly, for this case we shall write a balance equation for the maximum powers in the numerators of the terms from Eq. (7). There are 5 terms in Eq. (7). Each term has several powers of $\exp(\xi)$ in its numerator. From these several powers one is the maximum power for the corresponding term of Eq. (7). As the terms in Eq. (7) are 5, we have 5 maximum powers. We impose the balance equation constraint. The two largest of these 5 maximum powers must be equal (otherwise some parameters of the solution and eventually some parameters of the equation will be 0 which in the most of the cases is undesirable). The equality of the two largest powers leads to the balance equation.

The balance equation constraint leads to the following two possibilities for balance equations:

$$2LB + 3 = 2LB + 3 \quad \text{when} \quad H < 2 + \frac{2}{LB} \tag{9}$$

and

$$H = 2 + \frac{2}{LB}. \tag{10}$$

We note that L and H are integers and because of this the parameter B must have appropriate (even non-integer) values. Let us illustrate this point further by means of two small tables for the two possibilities for balance equations.

Let us now consider one examples. Let Eq. (10) be the balance equation and in addition $B = 1$ and $L = 1$. Then $H = 4$. Thus the equation we are going to solve is

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial}{\partial x}\left(\sum_{h=1}^{4} \alpha_h u^h\right) - \nu\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \gamma u\frac{\partial^3 u}{\partial x^3} = 0. \tag{11}$$

T a b l e 1

Several of the possible values of the parameters for the case when Eq. (9) is a balance equation

L	B	$2LB+3$	$H < 2 + \frac{2}{LB}$
1	1	5	$H < 4$
2	1	7	$H < 3$
1	1/2	4	$H < 6$
2	1/2	5	$H < 4$
...	...	...	...

Equation (7) becomes

$$u(x,t) = \sum_{l=0}^{1} A_l \left[\frac{\sum_{i=0}^{m} a_i \exp(i\xi)}{\sum_{j=0}^{n} b_j \exp(j\xi)} \right]^l, \quad \xi = kx + wt + \delta. \tag{12}$$

The substitution of Eq. (12) in Eq. (11) leads to a system of 5 nonlinear algebraic relationships among the parameters of the solution and the parameters of the equation. One solution of this system is

- $a > 0$, $b < 0$: $a_0 = 0$; $a_1 = a$; $b_0 = 1$, $b_1 = -b$;
- $a < 0$, $b > 0$: $a_0 = 0$; $a_1 = -a$; $b_0 = 1$; $b_1 = b$,

T a b l e 2

Several of the possible values of the parameters for the case when Eq. (10) is a balance equation

L	B	$H = 2 + \frac{2}{LB}$
1	1	4
2	1	3
1	1/2	6
2	1/2	4
3	1/3	4
...	...	...

and

$$\nu = -\frac{10A_1^2a^2\alpha_4 - 36aA_0b\alpha_4A_1 - 9A_1ab\alpha_3 + 18b^2\alpha_3A_0 + 6b^2\alpha_2 + 36b^2\alpha_4A_0^2}{k^2a^2b^2};$$

$$\gamma = \frac{4A_1^2a^2\alpha_4 - 12aA_0b\alpha_4A_1 - 3A_1ab\alpha_3 + 6b^2\alpha_3A_0 + 2b^2\alpha_2 + 12b^2\alpha_4A_0^2}{k^2a^2b^2};$$

$$w = \frac{1}{ka^2b^3}[A_1a(\alpha_4A_1^2a^2 - 8bA_1A_0\alpha_4a - A_1b\alpha_3a + 6b^2\alpha_3A_0 + b^2\alpha_2 + b^2\alpha_4A_0^2) -2A_0(3b^3A_0\alpha_3 - b^3\alpha_2 - 6b^3A_0^2\alpha_4)];$$

$$\alpha_1 = \frac{1}{k^2a^2b^3}[-4k^2A_1^2a^4A_0b\alpha_4 + 3k^2A_1a^3b^2\alpha_3A_0 + 6k^2A_1a^3b^2\alpha_4A_0^2 + 6b^3A_0^2\alpha_3 +2b^3A_0\alpha_2 + 12b^3A_0^3\alpha_4 - \alpha_4A_1^3a^3 + A_1^2b\alpha_3a^2 - A_1ab^2\alpha_2 + k^2A_1^3a^5\alpha_4 +8bA_1^2A_0\alpha_4a^2 - 6A_1ab^2\alpha_3A_0 - 18A_1ab^2\alpha_4A_0^2 - k^2A_1^2a^4b\alpha_3 + k^2A_1a^3b^2\alpha_2 -3k^2a^2A_0^2b^3\alpha_3 - 2k^2a^2A_0b^3\alpha_2 - 4k^2a^2A_0^3b^3\alpha_4]. \tag{13}$$

Thus the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2\partial t} + \frac{\partial u}{\partial x}\Bigg(\frac{1}{k^2a^2b^3}[-4k^2A_1^2a^4A_0b\alpha_4 + 3k^2A_1a^3b^2\alpha_3A_0 +6k^2A_1a^3b^2\alpha_4A_0^2 + 6b^3A_0^2\alpha_3 + 2b^3A_0\alpha_2 + 12b^3A_0^3\alpha_4 - \alpha_4A_1^3a^3 + A_1^2b\alpha_3a^2 -A_1ab^2\alpha_2 + k^2A_1^3a^5\alpha_4 + 8bA_1^2A_0\alpha_4a^2 - 6A_1ab^2\alpha_3A_0 - 18A_1ab^2\alpha_4A_0^2 -k^2A_1^2a^4b\alpha_3 + k^2A_1a^3b^2\alpha_2 - 3k^2a^2A_0^2b^3\alpha_3 - 2k^2a^2A_0b^3\alpha_2 - 4k^2a^2A_0^3b^3\alpha_4] +2\alpha_2u + 3\alpha_3u^2 + 4\alpha_4u^3\Bigg) - \nu\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \gamma u\frac{\partial^3 u}{\partial x^3} = 0 \tag{14}$$

has the solutions

$$u(\xi) = \frac{A_0 - \det\begin{bmatrix}A_0 & A_1\\ a & b\end{bmatrix}\exp(\xi)}{1 - b\exp(\xi)}, \quad a > 0, b < 0, \tag{15}$$

$$u(\xi) = \frac{A_0 + \det\begin{bmatrix}A_0 & A_1\\ a & b\end{bmatrix}\exp(\xi)}{1 + b\exp(\xi)}, \quad a < 0, b > 0, \tag{16}$$

where

$$\xi = kx + \frac{t}{ka^2b^3}[A_1a(\alpha_4A_1^2a^2 - 8bA_1A_0\alpha_4a - A_1b\alpha_3a + 6b^2\alpha_3A_0 +b^2\alpha_2 + b^2\alpha_4A_0^2) - 2A_0(3b^3A_0\alpha_3 - b^3\alpha_2 - 6b^3A_0^2\alpha_4)] + \delta \tag{17}$$

and det[...] denotes the determinant of the corresponding matrix.

3. Concluding remarks. For the nonlinear dynamics, chaos theory and for the nonlinear physics in general, the methods for obtaining exact analytical solutions of classes of nonlinear PDEs are of great interest. It seems, however, that some of these methods are more fundamental than other ones. In this paper we discuss the relations between the method of simplest equation and the exp-function method. On the basis of our experience gained by application of the method of simplest equation to various nonlinear PDEs, we consider a generalization (7) of the presentation of the exp-function method and then we demonstrate the obtaining of exact travelling wave solutions of a member of the class (8) of nonlinear PDEs.

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